Admission Control and Routing in Multi-priority Systems

by
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A dissertation submitted to the faculty of the University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Statistics and Operations Research.

Chapel Hill
2006

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ABSTRACT

Feng Chen: ADMISSION CONTROL AND ROUTING IN MULTI-PRIORITY SYSTEMS

(Under the direction of Professor Vidyadhar G. Kulkarni)

We consider a manufacturer that offers two types of prioritized warranties for its product. Type 1 warranty guarantees a shorter turnaround time than type 2 warranty. Hence items covered by type 1 warranty receive higher priority in repair service. When an item under warranty fails, the manufacturer sends it to one of several repair vendors for repair, who are under contracts to provide repair service for the manufacturer. The manufacturer pays each vendor a fixed fee per repair assignment. While an item is at the vendor under or awaiting repair, a linear holding cost is incurred by the vendor and a linear good-will cost is incurred by the manufacturer.

We first study the admission control problem for a single vendor that can either accept or reject an incoming repair assignment in order to maximize its own profit. We analyze the optimal control policies under three criteria: individual optimization, class optimization, and social optimization. By exploiting two proof methods, value iteration algorithm and sample path analysis, we prove that the optimal policy under each criterion has switching-curve structure. We also compare the optimal policies under the three criteria mentioned above and show that (i) the class-optimal policy accepts more high priority customers but fewer low priority customers than the socially optimal policy, which has interesting socioeconomic connotation, (ii) the individually optimal policy accepts more high priority customers than the class-optimal

policy, while it can accept either more or fewer low-priority customers than either of the other two optimal policies.

We then consider the warranty repair allocation problem which the manufacturer faces. The manufacturer's goal is to allocate the repair work in such a way that the total cost (including fixed cost and good-will cost) is minimized. The complexity of the problem makes the attempt to find the optimal policy very unlikely to succeed. Therefore, we turn our attention to heuristic routing procedures. We develop an effective and robust index-based policy by applying a single policy improvement step to a well-chosen static routing policy. We evaluate the index-based policy and compare it with other heuristics via simulation.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to my advisor, Professor Vidyadhar G. Kulkarni, for introducing me to the stochastic modeling area and for his priceless encouragement and guidance throughout the research. I am extremely grateful for his wise advice on my career path, which made the completion of the dissertation not only possible but also a truly enjoyable and rewarding experience.

I am very thankful to Professor Eylem Tekin for teaching great courses in Markov Decision Processes and Discrete Event Simulation, which were used as main tools in this research. Her many insightful suggestions were invaluable for the progress of the dissertation. I would also like to thank my committee members, Professor Amarjit Budhiraja, Professor Chuanshu Ji, Professor Jayashankar M. Swaminathan, and Professor Serhan Ziya, for their involvement and inspiring comments.

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α	discount rate
β	failure rate of each item
B_1	busy period for serving type 1 items initiated by a single type 1 item
B_2	busy period for serving both types of items initiated by a single type 2 item
C_{11}	expected holding cost incurred by the type 1 items during B_1
C_{12}	expected holding cost incurred by the type 1 items during S_2
G	Gini coefficient of initial state-independent allocation
h_i	holding cost rate of each class i customer
$H_1(x_1, x_2)$	expected total holding cost incurred by the system starting from initial state (x_1, x_2) until time $T_1(x_1)$
$H_2(x_1, x_2)$	expected total holding cost incurred by the system from time $T_1(x_1)$ until time $T(x_1, x_2)$
$H(x_1, x_2)$	expected total holding cost incurred by the system starting from initial state (x_1, x_2) until time $T(x_1, x_2)$
$I_{1j}(x_{1j}, x_{2j})$	class 1 index at vendor j in state (x_{1j}, x_{2j})
$I_{2j}(x_{1j}, x_{2j})$	class 2 index at vendor j in state (x_{1j}, x_{2j})
L_1^C	threshold for accepting class 1 customers under class optimization
L_1^I	threshold for accepting class 1 customers under individual optimization
$L_1^S(j)$	threshold for accepting class 1 customers under social optimization when there are j class 2 customers in the system
$L_2^C(i)$	threshold for accepting class 2 customers under class optimization when there are i class 1 customers in the system
$L_2^I(i)$	threshold for accepting class 2 customers under individual optimization when there are i class 1 customers in the system
$L_2^S(i)$	threshold for accepting class 2 customers under social optimization when there are i class 1 customers in the system

λ_i	arrival rate of class i customers
Λ	uniform rate
μ	repair service rate
ϕ_k	$\lambda_k W \beta$
$\phi_i(\alpha)$	$E(e^{-\alpha T} \| X(0) = i)$
p_{ki}	Bernoulli splitting probability of assigning a type k item to vendor i
p_{ki}^*	optimal Bernoulli splitting probability of assigning a type k item to vendor i
$\boldsymbol{p}_k$	$(p_{k1}, p_{k2}, \cdots, p_{kV})$
$\boldsymbol{p}_k^*$	$(p_{k1}^*, p_{k2}^*, \cdots, p_{kV}^*)$
$PP(\lambda)$	Poisson process with rate λ
r_i	reward for accepting a class i customer
S_2	the service completion time of a type 2 item accounting for interruptions from type 1 items. Thus if a type 2 item starts service at time 0, it will complete service at time S_2
$\tau_1(x_1)$	$E(T_1(x_1))$
$\tau(x_1, x_2)$	$E(T(x_1, x_2))$
T	$\min\{t \geq 0 : X(t) = 0\}$
$T_1(x_1)$	$\min\{t \geq 0 : X_1(t) = 0 \| X_1(0) = x_1\}$
$T(x_1, x_2)$	$\min\{t \geq 0 : X_1(t) = 0, X_2(t) = 0 \| X_1(0) = x_1, X_2(0) = x_2\}$
$v(i, j)$	minimum expected total discounted cost starting from state (i, j)
$v_n(i, j)$	nth-step value function generated by value iteration algorithm
$\mathcal{V}$	set of functions defined on $\{(i, j) : i \geq 0, j \geq 0\}$ that satisfy monotonicity in i and j, supermodularity, and diagonal dominance in j
$\bar{\mathcal{V}}$	set of functions defined on $\{(i, j) : i \geq 0, j \geq 0\}$ that satisfy monotonicity in i and j, supermodularity, diagonal dominance in i and j, and monotonicity in direction $(1, -1)$

V	total number of repair vendors
W	warranty length
$\lfloor x \rfloor$	largest integer less than or equal to x
$X(t)$	number of customer in an $M/M/1/k$ queue at time t
$X_k(t)$	number of type k items in the system at time t
$X(i,j)$	sojourn time of a class 1 customer who joins the system in state (i,j)
$Y(i,j)$	sojourn time of a class 2 customer who joins the system in state (i,j)
Z_k	service completion time of the kth class 2 customer accounting for interruptions from class 1 customers

Chapter 1

Introduction

1.1 Overview

Warranty has been playing an increasingly important role in product sales and services. In 2004, the 25 largest manufactures in the United States spent a total of $15 billion on warranty claims. Warranty claims processing consumed 2.5% $\sim$ 4.5% of revenues across all industries (Byrne [8]). It has been shown that warranty improvements can not only save cost but also boost revenues, enhance customer satisfaction and loyalty, and even drive up the product quality.

There has been a strong trend towards outsourcing various business operations in recent years, especially in the IT industry. According to IDC (a Framingham, Massachusetts-based market research firm), spending on IT outsourcing reached $56 billion in 2000 and $100 billion in 2005. As a major component of the manufacturing and retail industry, warranty repair services have experienced the rising outsourcing tide as well. Outsourcing warranty repairs offers the original equipment manufacturer the opportunity to reduce operating cost and capital investment, focus on their core business, increase speed to market, and faster customer response time.

Typically a manufacturer outsources repair work to several vendors, in which case the manufacturer faces the problem of how to distribute the workload among vendors in a cost-effective manner. The problem becomes more complicated in the presence of priorities. Priority issue arises when the manufacturer provides different types of warranties that specify different turnaround times. The warranty with shorter turnaround time is given to important customers (e.g. customers that make frequent or large purchases from the manufacturer), or sold to customers who are willing to pay more for a shorter repair time. To meet the specified turnaround times, products covered by a warranty that guarantees a shorter repair time are given higher priority in repair service. Hence, the manufacturer needs to solve a multi-priority warranty repair allocation problem.

We study two topics motivated by the problem mentioned above. The first topic is the admission control problem for a single vendor. We assume the failed items of each class (i.e., covered by each warranty) arrive at the vendor according to a Poisson process. The vendor can either accept or reject each arriving item with the objective of maximizing its own profit. The vendor receives a class-dependent reward each time it accepts an item and pays a holding cost at a class-dependent rate while an item is at the vendor. There is no penalty for rejecting an item. Costs and rewards are continuously discounted. We analyze the optimal admission control policy under three optimization criteria: individual optimization, class-optimization, and social optimization. Our primary interest is in showing structural properties of the optimal policies.

We first consider the case where the reward is generated at the time of joining the repair queue in Chapter 2. Using value iteration algorithm, we prove that the optimal policy is of threshold-type under each of the three optimization criteria mentioned above. We also compare the optimal policies under the three criteria and show that (i) the class-optimal policy accepts more class 1 customers but fewer class 2 customers than the socially optimal policy, which has interesting social connotation, (ii) the individually optimal policy accepts more class 1 customers than the class-optimal policy, while it can accept either more or fewer class 2 customers than either of the other two optimal policies. We then consider the case where the reward is generated at the time of service completion in Chapter 3. By applying sample path analysis, we show that the switching-curve structure property still holds for the optimal policy under each optimization criterion. We compare policies under different criteria numerically. The numerical results imply the same relationship between different criteria as proved for the first case.

The second topic is the dynamic routing problem for the manufacturer. Assume the life time of each item is exponential indepent of the warranty type. Each time an item covered under warranty fails, the manufacturer needs to decide which vendor to send the item for repair. The manufacturer pays a vendor-dependent fixed fee for each repair and incurs a good-will cost while an item is undergoing or waiting for repair. Given the complexity of the problem, trying to find the optimal solution is unrealistic. Hence we turn our attention to heuristic allocation procedures. In Chapter 4, we present four heuristics that are applicable to large problems, then evaluate and compare them using simulation. Among the four heuristics, the Generalized Join the Shortest Queue (GJSQ) policy is of our primary interest. The GJSQ policy is derived by applying a single policy improvement step to a judicious chosen initial static policy. We derive closed-form expressions for the GJSQ policy. The simulation results suggest that the GJSQ policy is robust and performs considerably better than the other heuristics.

1.2 Literature Review

There is an extensive literature on the subject of warranty. For a comprehensive reference, see Blischke and Murthy [7]. They discuss a variety of warranty policies including standard consumer product warranties such as the free replacement and pro rata, as well as warranties used in large volume or specialized transactions. Analytical models dealing with cost and optimization problems from both the manufacturer's and the buyer's point of view are developed. Methods of collecting and analyzing relevant data are also addressed. A literature review until 2002 is given by Murthy and Djamaludin [34]. For recent development, among others, see Dimitrov et al. [12], Yeh et al. [50], and Manna et al. [31].

1.2.1 Admission Control

Admission control for single class queueing systems is a well studied area. See Stidham [43] for a survey. The first quantitative model in this area is proposed by Naor [36], who studies an $M/M/1$ system with a single class of customers. He considers undiscounted reward and cost and the objective is to maximize the long-run average net reward per unit time. Naor considers only critical-number policies and shows that $n_S \leq n_I$, where n_S and n_I are the critical numbers for social optimization and individual optimization, respectively. An incoming customer is accepted if the number of existing customer is less than the critical number and rejected otherwise. Yechiali [48] [49] proves that for $GI/M/1$, $GI/M/s$ systems the socially optimal policy has critical-number form. Thus Naor's restriction to critical-number policies is without loss of generality.

Naor's result has been generalized by many authors. Among others, Knudsen [22] considers an $M/M/s$ queue with state-dependent net benefit. Lippman and Stidham [28] study a birth-death process with general departure rate, random reward, with or without discounting and for a finite or infinite time horizon. Stidham [42] considers a $GI/M/1$ queue with random reward and general holding cost, with or without discounting. For other models of admission control problem for single-class queues, see Adiri and Yechiali [1], Stidham and Weber [44], and Rykov [40].

Admission control for multi-class queueing systems is another important research area. Models in this area can be classified into two categories based on whether or not service is prioritized based on class. In models without priorities, different classes are distinguished by different arrival rates, service rates, rewards, holding costs, etc. For papers in this category, among others, see Miller [33], Blanc and de Waal [6], Kulkarni and Tedijanto [25], and Nair and Bapma [35]. Among papers that consider service priorities, Mendelson and Whang [32] study a priority pricing problem for a multi-class $M/M/1$ queueing system, where each customer decides by himself whether or not to join the system and, if join, at what priority level. Hassin [19] studies a bidding mechanism for determining priorities in a $GI/M/1$ queue without balking. Ha [18] considers the production control problem in a make-to-stock production system with two prioritized customer classes.

To the best of our knowledge, the admission control problem for a multi-class queue with predetermined priorities and the objective of minimizing expected total discounted cost has not been studied. Besides the widely used individual optimization and social optimization, we propose a new optimization criterion: class optimization. Using two proof methods: value iteration algorithm and sample path analysis, we show that the optimal policies have threshold-type structure. We also compare between different optimal policies.

1.2.2 Warranty Repair Routing

We categorize warranty repair routing problems from the following four aspects.

(i) Based on the priority levels, we have either *single-priority problems* or *multi-priority problems*. In the single-priority case, the repair service at each vendor is provided on a first-in, first-served basis. In the multi-priority case, the repair service is provided based on a predetermined priority policy.

(ii) Based on the number of items under warranty, we have either *fixed-population problems*, or *variable-population problems.* Fixed-population problem arises when we are dealing with warranty repairs for a batch of items sold at once, in which case no items enter or leave the warranty population of interest during the warranty period. More often, the items are sold in a continuous fashion. Thus the number of items under warranty increases when a new sale occurs and decreases when the warranty expires on an existing item, in which case we have a variable-population problem.

(iii) Based on the assignment rule, we can use either *assign-at-purchase policies* or *assign-at-failure policies.* The former requires an item to be assigned to a vendor at the time of purchase and sent to that vendor for repair each time it fails. This can be done by printing the repair vendor's phone number on the warranty card and instructing customers to call that number for repair services. The latter allows the items to be assigned to different vendors at the time of failure. In this case, a routing center's phone number is printed on the warranty card. A repair vendor's information is provided when the customer calls with a request for repair.

(iv) Based on the available information, the routing policy we use can be either *state-independent*, *partially state-dependent*, or *fully state-dependent.* State-independent policies do not use any real-time information of the system, i.e., the same rule is applied to every assignment. Partially state-dependent policies use only the real-time information of the warranty population, which includes the number of items under each type of warranty and the remaining warranty length of each item. This information can be easily collected by keeping a record of the purchases made in the past W time, where W is the warranty length. If the warranty periods are assumed to be i.i.d. exponential random variables, then only the warranty population size is necessary. Fully state-dependent policies use real-time information of both the warranty population and the vendors. Real-time information of vendors means the number of items at each vendor at the time of each failure. Collecting this information requires real-time communication between the manufacturer and the vendors, which may need a more complicated information system and cost extra.

The warranty repair allocation problem has the simplest structure when considering single-priority and fixed-population. In this case, the assign-at-purchase model reduces to a resource allocation problem with separable objective function. Note that only state-independent policies are applicable in the assign-at-purchase model. This problem has been extensively studied in the literature. When the objective is convex, a simple greedy algorithm first proposed by Gross [17] can be used to solve the problem optimally. See Ibaraki and Katoh [21] for a comprehensive reference for the resource allocation problems. Opp et al. [37] discuss the application of the greedy algorithm to the warranty repair allocation problem. Ding and Glazebrook [13] consider a goodwill cost model that takes explicit account of the delays experienced by customers. They show that simple greedy approaches work well. The assign-at-failure model for single-priority and fixed-population problem is studied by Opp et al. [37]. They argue that optimally solving real-life size problems is numerically intractable. They develop index-based, fully state-dependent heuristic policies to find near-optimal solutions.

When priorities are considered, the objective function is no longer separable. Buczkowski et al. [9] study the assign-at-purchase model for multi-priority, fixed-population problems. They formulate the problem as a minimum cost network flow problem and provide an efficient algorithm to solve it.

We are interested in the multi-priority, variable-population problem, and the assign-at-failure policies. Given the difficulty of the problem even without considering priority and finite constant warranty length (see Opp et al. [37]), seeking the optimal solution is very unlikely to be successful. Hence we focus on constructing heuristic policies.

We first simplify the problem by assuming that the number of functioning items under warranty of each type is a constant. Therefore, failures occur according to Poisson processes. The original warranty repair allocation problem reduces to the problem of routing items arriving according to independent Poisson streams to several vendors where service is provided according to a fixed priority policy. A general model of this situation is studied by Ansell et al. [3]. They develop an index-based dynamic routing heuristic by applying a single policy improvement step to an initial static policy (see also Krishnan [23] and Tijms [45] for this approach). They name the resulting index-based heuristic "Generalized Join the Shortest Queue"(GJSQ) policy.

The simplified version of our problem is a special case of the model studied by Ansell et al. [3] (we consider two generic classes and no dedicated classes), except that we allow a station-dependent fixed cost per assignment, which is not considered by Ansell et al. [3]. We adapt their approach and derive tractable closed-form expressions for the indices, which are given as a solution to an infinite set of recursive equations in Ansell et al. [3]. We evaluate the GJSQ policy and compare it with three other heuristics using simulation. The simulation results show that, although the GJSQ policy is derived based on a simplified model, it works well on the original problem and outperforms the other heuristics.

Chapter 2

Admission Control: Value Iteration Approach

2.1 Problem Description

We study the admission control problem at a single vendor in this chapter. We model the single vendor under consideration as an $M/M/1$ queueing system serving two classes of customers. Class 1 customers have preemptive resume priority over class 2 customers. Within each class, the service is provided on a first-come, first-served basis. Class i customers arrive according to a Poisson process with parameter λ_i, $i = 1, 2$. Each customer requires an i.i.d. $\exp(\mu)$ service time (same for both classes). The system is controlled by accepting or rejecting arriving customers. There is a reward of r_i associated with accepting a class i customer. An accepted class i customer generates a waiting cost of h_i per unit time spent in the system. All rewards and costs are continuously discounted with rate $\alpha > 0$. The goal is to minimize the expected total discounted net cost.

Priority issue arises in many other queueing systems. For example, internet traffic protocols assign higher priority to data packages that require real-time transmission (e.g. live audio and video) and lower priority to delay-insensitive packages (e.g. e-mails and file transmission). Service queues may give VIP customers higher priority over ordinary customers. In hospitals, patients in critical conditions receive higher priority in treatment over non-critical patients. Admission control problem in these kinds of multi-priority queues can be modeled by the framework presented here.

We analyze the optimal control policies for such a system under 3 criteria: individual optimization, class optimization, and social optimization. Under individual optimization, each customer obtains the reward and pays the waiting cost by himself. A customer makes decision based on the objective of minimizing his own expected total discounted net cost. Under class optimization, there is a controller for each class. The controller of class i obtains the reward and pays the waiting cost generated by each class i customer. He decides whether to accept an arriving class i customer or not based on the objective of minimizing the expected total discounted net cost incurred by all class i customers. Under social optimization, there is a single controller for the whole system. The system controller obtains the reward and pays the waiting cost generated by every customer. He decides whether to accept an arriving customer or not based on the objective of minimizing the expected total discounted net cost incurred by all customers.

2.2 Individual Optimization

We consider individual optimization in this section. Clearly, the individually optimal policy for an arriving customer is to join the system if and only if his expected discounted net cost is less than or equal to zero.

Denote the system state by (i, j), where i is the number of class 1 customers in the system and j is the number of class 2 customers in the system. We need the following lemma to derive the main result in Theorem 1.

Lemma 1. *Let $X(t)$ be the number of customers in a $M/M/1/k$ queue at time t with arrival rate λ and service rate μ. Let $T = \min\{t \geq 0 : X(t) = 0\}$ and define $\phi_i(\alpha) = E(e^{-\alpha T}|X(0) = i)$. Then, $\phi_i(\alpha)$ is given by*

$$\phi_i(\alpha) = \frac{u_1^i u_2^{k-1}(u_2(\alpha+\mu)-\mu) - u_2^i u_1^{k-1}(u_1(\alpha+\mu)-\mu)}{u_2^{k-1}(u_2(\alpha+\mu)-\mu) - u_1^{k-1}(u_1(\alpha+\mu)-\mu)}, \ i = 0, \ldots, k, \tag{2.1}$$

where

$$\begin{aligned} u_1 &= \tfrac{1}{2\lambda_1}(\alpha + \lambda_1 + \mu + \sqrt{(\alpha+\lambda_1+\mu)^2 - 4\lambda_1\mu}), \\ u_2 &= \tfrac{1}{2\lambda_1}(\alpha + \lambda_1 + \mu - \sqrt{(\alpha+\lambda_1+\mu)^2 - 4\lambda_1\mu}). \end{aligned} \tag{2.2}$$

Proof. $\{X(t), t \geq 0\}$ is a birth-death process on state space $S = \{0, 1, \ldots, k\}$. By Theorem 6.21 of Kulkarni [24], $\{\phi_i(\alpha)\}$ is the solution to

$$\begin{aligned} \phi_0(\alpha) &= 1, \\ \mu\phi_{i-1}(\alpha) - (\alpha+\lambda_1+\mu)\phi_i(\alpha) + \lambda_1\phi_{i+1}(\alpha) &= 0, \quad i = 1, 3, \ldots, k-1, \\ \mu\phi_{k-1}(\alpha) - (\alpha+\mu)\phi_k(\alpha) &= 0. \end{aligned} \tag{2.3}$$

Solving the above system of equations yields (2.1). □

Theorem 1. *Under the individual optimization criterion, an arriving class 1 customer who sees the system in state (i, j) joins the queue if and only if $i < L_1^I$, where*

$$L_1^I = \begin{cases} \infty, & \text{if } h_1 \leq \alpha r_1 \\ \lfloor \log(1 - \frac{\alpha r_1}{h_1}) / \log \frac{\mu}{\mu+\alpha} \rfloor, & \text{if } h_1 > \alpha r_1. \end{cases} \tag{2.4}$$

An arriving class 2 customer who sees the system in state (i, j) joins the queue if and only if $j < L_2^I(i)$, where

$$L_2^I(i) = \begin{cases} \infty, & \text{if } h_2 \leq \alpha r_2 \\ \lfloor \log \frac{h_2 - \alpha r_2}{h_2 \phi_i(\alpha)} / \log \beta \rfloor, & \text{if } h_2 > \alpha r_2, \ i \leq L_1^I \\ \lfloor (\log \frac{h_2 - \alpha r_2}{h_2 \phi_{L_1^I}(\alpha)} + (i - L_1^I)(\log \frac{\mu+\alpha}{\mu})) / \log \beta \rfloor, & \text{if } h_2 > \alpha r_2, \ i > L_1^I, \end{cases} \tag{2.5}$$

where $\phi_i(\alpha)$ is given in (2.1), $\beta = \frac{\mu}{\alpha+\mu+\lambda_1(1-\phi_1(\alpha))}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x. Furthermore, $L_2^I(i)$ is decreasing in i.

Proof. First consider class 1 customers. Denote the sojourn time of a class 1 customer who joins the system in state (i, j) by $X(i, j)$. Since class 1 customers have preemptive priority over class 2 customers, we have

$$X(i, j) = X_1 + X_2 + \cdots + X_{i+1},$$

where $X_k, k = 1, 2, \ldots, i+1$ are i.i.d. $\exp(\mu)$ service times. So the class 1 customer's expected total discounted cost is

$$E(\int_0^{X(i,j)} h_1 e^{-\alpha t} dt) = \frac{h_1}{\alpha}(1 - (\frac{\mu}{\mu+\alpha})^{i+1}).$$

Therefore, he joins the queue if and only if

$$\frac{h_1}{\alpha}(1 - (\frac{\mu}{\mu+\alpha})^{i+1}) \leq r_1, \tag{2.6}$$

which is equivalent to $i < L_1^I$, where L_1^I is defined in (2.4).

Now consider class 2 customers. Denote the sojourn time of a class 2 customer who joins the system in state (i, j) by $Y(i, j)$. We can decompose $Y(i, j)$ into 3 periods. Period 1, denoted by T_1, is the time period for serving the first $i - L_1^I$ class 1 customers, if $i > L_1^I$. Period 1 has length 0 if $i \leq L_1^I$. Note that no class 1 arrivals will be accepted during this period. Period 2, denoted by T_2, is the server's busy period for serving the remaining class 1 customers and the class 1 customers joining the system during this period, which ends when the first class 2 customer starts receiving service. Period 3, denoted by T_3, is the time period for serving the $j+1$ class 2 customers and the class 1 customers joining the system during this period.

Consider period T_1 first. If $i \leq L_1^I$, T_1 has length 0, thus $E(e^{-\alpha T_1}) = 1$. If $i > L_1^I$,

T_1 is the sum of $i - L_1^I$ i.i.d. $\exp(\mu)$ service times. Thus $E(e^{-\alpha T_1}) = (\frac{\mu}{\alpha+\mu})^{i-L_1^I}$.

From Lemma 1 we know the LST of T_2 is given by (2.1) with $k = L_1^I$.

Consider period T_3. $T_3 = \sum_{k=1}^{j+1} Z_k$, where Z_k is the time period for serving the kth class 2 customer and the class 1 customers joining the system during this period. Let $\beta = E(e^{-\alpha Z_1})$. Using first-step analysis, one can show that β satisfies

$$\beta = \frac{\mu+\lambda_1}{\alpha+\mu+\lambda_1}(\frac{\mu}{\mu+\lambda_1} + \frac{\lambda_1}{\mu+\lambda_1}\phi_1(\alpha)\beta).$$

Solving for β, we have

$$\beta = \frac{\mu}{\alpha+\mu+\lambda_1(1-\phi_1(\alpha))}.$$

Since $\{Z_k\}$ are i.i.d., we have

$$E(e^{-\alpha T_3}) = (E(e^{-\alpha Z_1}))^{j+1} = \beta^{j+1}.$$

Thus

$$E(e^{-\alpha Y(i,j)}) = E(e^{-\alpha T_1})E(e^{-\alpha T_2})E(e^{-\alpha T_3}) = (\frac{\mu}{\alpha+\mu})^{\max\{0,i-L_1^I\}}\phi_{\min\{i,L_1^I\}}(\alpha)\beta^{j+1}.$$

Therefore, the expected total discounted cost for a class 2 customer joining the system in state (i, j) is

$$E(\int_0^{Y(i,j)} h_2 e^{-\alpha t} dt) = \frac{h_2}{\alpha}(1 - (\frac{\mu}{\alpha+\mu})^{\max\{0,i-L_1^I\}}\phi_{\min\{i,L_1^I\}}(\alpha)\beta^{j+1}).$$

He will join the system if and only if

$$\frac{h_2}{\alpha}(1 - (\frac{\mu}{\alpha+\mu})^{\max\{0,i-L_1^I\}}\phi_{\min\{i,L_1^I\}}(\alpha)\beta^{j+1}) \le r_2,$$

which is equivalent to $j < L_2^I(i)$, where $L_2^I(i)$ is defined in (2.5).

Since T is stochastically increasing in i, $\phi_i(\alpha)$ is decreasing in i. Thus $L_2^I(i)$ is decreasing in i. $\square$

2.3 Class Optimization

We consider class optimization in this section. There is a controller for each class. The controller of class i decides whether to accept an arriving class i customer or not based on the objective of minimizing the expected total discounted net cost incurred by all class i customers, $i = 1, 2$.

Consider the optimal policy for the controller of class 1 first. This is the standard single-class admission control problem studied by many authors. Among others, Stidham [42] considers a $GI/M/1$ queue with random rewards and general holding cost and shows that the optimal policy is of critical-number form. As a special case, we have

Theorem 2. *The optimal policy for the controller of class 1 is a threshold policy, i.e., there exists a constant L_1^C such that an arriving class 1 customer is accepted if and only if $i < L_1^C$.*

Now consider the optimal policy for the controller of class 2. Assume that the controller of class 1 applies his optimal policy and the controller of class 2 knows that. Let $v(i,j)$ be the minimum expected total discounted cost for the controller of class 2 with initial state (i,j). Following Lippman [27], we uniformize the process by defining the uniform rate $\Lambda = \lambda_1 + \lambda_2 + \mu$. Assuming, without loss of generality, $\Lambda + \alpha = 1$, the optimality equations can be written as

$$v(i,j) = Tv(i,j) = C(j) + \lambda_1 T_1 v(i,j) + \lambda_2 T_2 v(i,j) + \mu T_3 v(i,j), \tag{2.7}$$

where

$$C(j) = h_2 j, \tag{2.8}$$

$$T_1 v(i,j) = \begin{cases} v(i+1,j), & i < L_1^C \\ v(i,j), & i \geq L_1^C, \end{cases} \tag{2.9}$$

$$T_2 v(i,j) = \min\{-r_2 + v(i,j+1), v(i,j)\}, \tag{2.10}$$

and

$$T_3 v(i,j) = \begin{cases} v(i-1,j), & i \geq 1, j \geq 0 \\ v(0,j-1), & i = 0, j \geq 1 \\ v(0,0), & i = 0, j = 0. \end{cases} \tag{2.11}$$

Let $\mathcal{V}$ be the set of functions such that if $v \in \mathcal{V}$, then

- v is monotonically increasing in i, i.e.,

$$v(i,j) \leq v(i+1,j), \tag{2.12}$$

- v is monotonically increasing in j, i.e.,

$$v(i,j) \leq v(i,j+1), \tag{2.13}$$

- v is supermodular, i.e.,

$$v(i,j+1) + v(i+1,j) \leq v(i,j) + v(i+1,j+1), \tag{2.14}$$

- v is diagonally dominant in j, i.e.,

$$v(i,j+1) + v(i+1,j+1) \leq v(i+1,j) + v(i,j+2). \tag{2.15}$$

It is worth noting that if $v \in \mathcal{V}$, then v is convex in j, i.e.,

$$v(i, j+1) - v(i, j) \leq v(i, j+2) - v(i, j+1). \tag{2.16}$$

This follows by adding inequalities (2.14) and (2.15).

We have the following properties of the operators T_1, T_2, and T_3.

Lemma 2. *If $v \in \mathcal{V}$, then $T_1 v \in \mathcal{V}$.*

Proof.

(a) For (2.12), if $i \leq L_1^C - 2$, then

$$T_1 v(i, j) = v(i+1, j) \leq v(i+2, j) = T_1 v(i+1, j).$$

If $i = L_1^C - 1$, then

$$T_1 v(i, j) = v(i+1, j) = T_1 v(i+1, j).$$

If $i \geq L_1^C$, then

$$T_1 v(i, j) = v(i, j) \leq v(i+1, j) = T_1 v(i+1, j).$$

(b) For (2.13), if $i \leq L_1^C - 1$, then

$$T_1 v(i, j) = v(i+1, j) \leq v(i+1, j+1) = T_1 v(i, j+1).$$

If $i \geq L_1^C$, then

$$T_1 v(i, j) = v(i, j) \leq v(i, j+1) = T_1 v(i, j+1).$$

(c) For (2.14), if $i \leq L_1^C - 2$, then

$$\begin{aligned} T_1 v(i, j+1) + T_1 v(i+1, j) &= v(i+1, j+1) + v(i+2, j) \\ &\leq v(i+1, j) + v(i+2, j+1) \\ &= T_1 v(i, j) + T_1 v(i+1, j+1), \end{aligned}$$

where the inequality follows from (2.14) with i replaced by $i+1$.

If $i = L_1^C - 1$, then

$$\begin{aligned} T_1 v(i, j+1) + T_1 v(i+1, j) &= v(i+1, j+1) + v(i+1, j) \\ &= T_1 v(i, j) + T_1 v(i+1, j+1). \end{aligned}$$

If $i \geq L_1^C$, then

$$\begin{aligned} T_1 v(i, j+1) + T_1 v(i+1, j) &= v(i, j+1) + v(i+1, j) \\ &\leq v(i, j) + v(i+1, j+1) \\ &= T_1 v(i, j) + T_1 v(i+1, j+1). \end{aligned}$$

(d) For (2.15), if $i \leq L_1^C - 2$, then

$$\begin{aligned} T_1 v(i, j+1) + T_1 v(i+1, j+1) &= v(i+1, j+1) + v(i+2, j+1) \\ &\leq v(i+2, j) + v(i+1, j+2) \\ &= T_1 v(i+1, j) + T_1 v(i, j+2), \end{aligned}$$

where the inequality follows from (2.15) with i replaced by $i+1$.

If $i = L_1^C - 1$, then

$$\begin{aligned} T_1 v(i, j+1) + T_1 v(i+1, j+1) &= v(i+1, j+1) + v(i+1, j+1) \\ &\leq v(i+1, j) + v(i+1, j+2) \\ &= T_1 v(i+1, j) + T_1 v(i, j+2), \end{aligned}$$

where the inequality follows from (2.16) with i replaced by $i+1$.

If $i \geq L_1^C$, then

$$\begin{aligned} T_1 v(i, j+1) + T_1 v(i+1, j+1) &= v(i, j+1) + v(i+1, j+1) \\ &\leq v(i+1, j) + v(i, j+2) \\ &= T_1 v(i+1, j) + T_1 v(i, j+2). \end{aligned}$$

□

Lemma 3. *If $v \in \mathcal{V}$, then $T_2 v \in \mathcal{V}$.*

Proof.

(a) For (2.12), denote by a the minimizing action in $T_2 v(i+1, j)$, where action 0 (1) refers to rejecting (accepting) a customer, i.e., $T_2 v(i+1, j) = \min\{-r_2 + v(i+1, j+1), v(i+1, j)\} = v(i+1, j)$, if $a = 0$, and $T_2 v(i+1, j) = -r_2 + v(i+1, j+1)$, if $a = 1$.

If $a = 0$, then

$$T_2 v(i, j) = \min\{-r_2 + v(i, j+1), v(i, j)\} \leq v(i, j) \leq v(i+1, j) = T_2 v(i+1, j).$$

If $a = 1$, then

$$T_2 v(i, j) \leq -r_2 + v(i, j+1) \leq -r_2 + v(i+1, j+1) = T_2 v(i+1, j).$$

(b) For (2.13), the proof is similar to (a).

(c) For (2.14), denote by a_1 (a_2) the minimizing action in $T_2v(i,j)$ ($T_2v(i{+}1,j{+}1)$).

If $a_1 = a_2 = 0$, then

$$
\begin{aligned}
& T_2v(i,j+1) + T_2v(i+1,j) \\
= \;& \min\{-r_2 + v(i,j+2), v(i,j+1)\} + \min\{-r_2 + v(i+1,j+1), v(i+1,j)\} \\
\leq \;& v(i,j+1) + v(i+1,j) \leq v(i,j) + v(i+1,j+1) = T_2v(i,j) + T_2v(i+1,j+1),
\end{aligned}
$$

where the second inequality follows from (2.14).

The case where $a_1 = a_2 = 1$ can be proved similarly.

If $a_1 = 1, a_2 = 0$, then

$$
\begin{aligned}
T_2v(i,j+1) + T_2v(i+1,j) &\leq v(i,j+1) - r_2 + v(i+1,j+1) \\
&= T_2v(i,j) + T_2v(i+1,j+1).
\end{aligned}
$$

If $a_1 = 0, a_2 = 1$, following the convention that an arriving customer is accepted when the system performance is indifferent between accepting and rejecting this customer, we have

$$
v(i,j) < -r_2 + v(i,j+1), \quad -r_2 + v(i+1,j+2) \leq v(i+1,j+1).
$$

The sum of the these two inequalities gives us

$$
v(i,j) + v(i+1,j+2) < v(i,j+1) + v(i+1,j+1). \tag{2.17}
$$

Replacing j by $j+1$ in (2.14), we get

$$v(i, j+2) + v(i+1, j+1) \leq v(i, j+1) + v(i+1, j+2). \tag{2.18}$$

Summing up (2.14), (2.15) and (2.18), we get

$$v(i, j+1) + v(i+1, j+1) \leq v(i, j) + v(i+1, j+2),$$

which is a contradiction to (2.17). Therefore, the case where $a_1 = 0, a_2 = 1$ does not exist.

(d) For (2.15), denote by a_1 (a_2) the minimizing action in $T_2v(i+1, j)$ ($T_2v(i, j+2)$).

If $a_1 = a_2 = 0$, then

$$\begin{aligned}
& T_2v(i, j+1) + T_2v(i+1, j+1) \\
= \ & \min\{-r_2 + v(i, j+2), v(i, j+1)\} + \min\{-r_2 + v(i+1, j+2), v(i+1, j+1)\} \\
\leq \ & v(i, j+1) + v(i+1, j+1)\} \\
\leq \ & v(i+1, j) + v(i, j+2) = T_2v(i+1, j) + T_2v(i, j+2),
\end{aligned}$$

where the second inequality follows from (2.15).

The case where $a_1 = a_2 = 1$ can be proved similarly.

If $a_1 = 1, a_2 = 0$, then

$$\begin{aligned}
& T_2v(i, j+1) + T_2v(i+1, j+1) \\
\leq \ & -r_2 + v(i, j+2) + v(i+1, j+1) = T_2v(i+1, j) + T_2v(i, j+2).
\end{aligned}$$

If $a_1 = 0, a_2 = 1$, then

$$v(i+1,j) < -r_2 + v(i+1,j+1), \quad -r_2 + v(i,j+3) \leq v(i,j+2).$$

The sum of the above two inequalities gives us

$$v(i+1,j) + v(i,j+3) < v(i+1,j+1) + v(i,j+2). \tag{2.19}$$

Replacing j by $j+1$ in (2.15), we have

$$v(i,j+2) + v(i+1,j+2) \leq v(i+1,j+1) + v(i,j+3). \tag{2.20}$$

Summing up (2.15), (2.18), and (2.20), we get

$$v(i+1,j+1) + v(i,j+2) \leq v(i+1,j) + v(i,j+3),$$

which is a contradiction to (2.19). Therefore the case where $a_1 = 0, a_2 = 1$ does not exist.

□

Lemma 4. *If $v \in \mathcal{V}$, then $T_3 v \in \mathcal{V}$.*

Proof.

(a) For (2.12), if $i \geq 1, j \geq 0$, then

$$T_3 v(i,j) = v(i-1,j) \leq v(i,j) = T_3 v(i+1,j).$$

If $i = 0, j \geq 1$, then

$$T_3 v(0,j) = v(0,j-1) \leq v(0,j) = T_3 v(1,j).$$

If $i = 0, j = 0$, then

$$T_3v(0,0) = v(0,0) = T_3v(1,0).$$

(b) For (2.13), the proof is similar to (a).

(c) For (2.14), if $i \geq 1, j \geq 0$, then

$$\begin{aligned} & T_3v(i,j+1) + T_3v(i+1,j) = v(i-1,j+1) + v(i,j) \\ \leq\ & v(i-1,j) + v(i,j+1) = T_3v(i,j) + T_3v(i+1,j+1), \end{aligned}$$

where the inequality follows from (2.14) with i replaced by $i-1$.

If $i = 0, j \geq 1$, then

$$\begin{aligned} & T_3v(0,j+1) + T_3v(1,j) = v(0,j) + v(0,j) \\ \leq\ & v(0,j-1) + v(0,j+1) = T_3v(0,j) + T_3v(1,j+1), \end{aligned}$$

where the inequality follows from (2.16) with j replaced by $j-1$ and $i = 0$.

If $i = 0, j = 0$, then

$$\begin{aligned} & T_3v(0,1) + T_3v(1,0) = v(0,0) + v(0,0) \\ \leq\ & v(0,0) + v(0,1) = T_3v(0,0) + T_3v(1,1). \end{aligned}$$

(d) For (2.15), if $i \geq 1, j \geq 0$, then

$$\begin{aligned} & T_3v(i,j+1) + T_3v(i+1,j+1) = v(i-1,j+1) + v(i,j+1) \\ \leq\ & v(i,j) + v(i-1,j+2) = T_3v(i+1,j) + T_3v(i,j+2), \end{aligned}$$

where the inequality follows from (2.15) with i replaced by $i-1$.

If $i = 0, j \geq 1$, then

$$T_3v(0, j+1) + T_3v(1, j+1) = v(0, j) + v(0, j+1) = T_3v(1, j) + T_3v(0, j+2).$$

If $i = 0, j = 0$, then

$$T_3v(0, 1) + T_3v(1, 1) = v(0, 0) + v(0, 1) = T_3v(1, 0) + T_3v(0, 2).$$

□

The above lemmas lead to the following theorem.

Theorem 3. *The optimal value function $v \in \mathcal{V}$.*

Proof. Let $v_0(i, j) = 0$, $\forall (i, j) \in S$, and define, for $n \geq 0$, $v_{n+1}(i, j) = C(j) + \lambda_1 T_1 v_n(i, j) + \lambda_2 T_2 v_n(i, j) + \mu T_3 v_n(i, j)$. Since $\alpha > 0$, we know that $v_n \to v$ as $n \to \infty$. (See Theorem 6.3.1 of Puterman [39].)

It is easy to see that $C(j) \in \mathcal{V}$. Lemma 2, 3, 4 show that if $v_n \in \mathcal{V}$ then $T_i v_n \in \mathcal{V}$ for $i = 1, 2, 3$. Clearly $v_0 \in \mathcal{V}$ and the above observation yields that if $v_n \in \mathcal{V}$ then $v_{n+1} \in \mathcal{V}$. Hence, by induction, $v_n \in \mathcal{V}$ for all n. Therefore, by taking limits, $v \in \mathcal{V}$, thus proving the theorem □

Now we are ready to prove the structural properties of the class-optimal policy for class 2 customers.

Theorem 4. *The optimal policy for the controller of class 2 is characterized by a monotonically decreasing switching curve, i.e., for each $i \geq 0$, there exists a threshold $L_2^C(i)$, such that a class 2 arrival in state (i, j) is accepted if and only if $j < L_2^C(i)$. Furthermore, $L_2^C(i)$ is monotonically decreasing in i.*

Proof. From (2.10) we can see that a class 2 arrival in state (i, j) is accepted if and only if

$$v(i, j+1) - v(i, j) \leq r_2. \tag{2.21}$$

Let

$$L_2^C(i) = \min\{j : v(i, j+1) - v(i, j) > r_2\}.$$

By using property (2.16), one can show that condition (2.21) is equivalent to $j < L_2^C(i)$.

For $i_1 \leq i_2$, we have $v(i_2, j+1) - v(i_2, j) \geq v(i_1, j+1) - v(i_1, j)$, which follows from property (2.14). By definition of $L_2^C(i_1)$, we have $v(i_1, L_2^C(i_1)+1) - v(i_1, L_2^C(i_1)) > r_2$, so $v(i_2, L_2^C(i_1)+1) - v(i_2, L_2^C(i_1)) > r_2$. By definition of $L_2^C(i_2)$, we have $L_2^C(i_1) \geq L_2^C(i_2)$. Thus, $L_2^C(i)$ is decreasing in i. $\square$

2.4 Social Optimization

We consider social optimization in this section. There is a single controller for the whole system, he earns the rewards and pays the holding costs generated by all customers. Let $v(i, j)$ be the minimum expected total discounted cost for the system controller with initial state (i, j). Using uniform rate $\Lambda = \lambda_1 + \lambda_2 + \mu$, and assuming, without loss of generality, $\Lambda + \alpha = 1$, the optimality equations can be written as

$$v(i, j) = \bar{T}v(i, j) = \bar{C}(i, j) + \lambda_1 \bar{T}_1 v(i, j) + \lambda_2 T_2 v(i, j) + \mu T_3 v(i, j), \tag{2.22}$$

where

$$\bar{C}(i, j) = h_1 i + h_2 j,$$

$$\bar{T}_1 v(i, j) = \min\{-r_1 + v(i+1, j), v(i, j)\}, \tag{2.23}$$

T_2 and T_3 are as defined in (2.10) and (2.11), respectively.

Let $\bar{\mathcal{V}}$ be the set of functions such that if $v \in \bar{\mathcal{V}}$, then v satisfies (2.12) - (2.15), and

- v is diagonally dominant in i, i.e.,

$$v(i+1,j) + v(i+1,j+1) \le v(i,j+1) + v(i+2,j), \tag{2.24}$$

- v is increasing in the direction of $(1,-1)$, i.e.,

$$v(i,j+1) \le v(i+1,j). \tag{2.25}$$

Notice that if $v \in \bar{\mathcal{V}}$, then v is convex in i, i.e.,

$$v(i+1,j) - v(i,j) \le v(i+2,j) - v(i+1,j). \tag{2.26}$$

This follows by adding inequalities (2.14) and (2.24).

We have the following lemmas.

Lemma 5. *If $v \in \bar{\mathcal{V}}$, then $\bar{T}_1 v \in \bar{\mathcal{V}}$.*

Proof.

(a) For (2.12) and (2.13), the proofs are similar to part (a) of the proof of Lemma 3.

(b) Since (2.14) is symmetric with respect to i and j, the proof of $\bar{T}_1$ preserving (2.14) is the same as part (c) of the proof of Lemma 3 with r_2 replaced by r_1 and i,j interchanged, e.g., replace term $v(i+1,j)$ by $v(i,j+1)$.

(c) For (2.15), denote by a_1 (a_2) the minimizing action in $\bar{T}_1 v(i+1,j)$ ($\bar{T}_1 v(i,j+2)$).

If $a_1 = a_2 = 0$, then

$$\begin{aligned}
& \bar{T}_1 v(i, j+1) + \bar{T}_1 v(i+1, j+1) \\
\leq\ & v(i, j+1) + v(i+1, j+1)\} \\
\leq\ & v(i+1, j) + v(i, j+2) = \bar{T}_1 v(i+1, j) + \bar{T}_1 v(i, j+2),
\end{aligned}$$

where the second inequality follows from (2.15).

The case where $a_1 = a_2 = 1$ can be proved similarly.

If $a_1 = 1, a_2 = 0$, then

$$\begin{aligned}
& \bar{T}_1 v(i, j+1) + \bar{T}_1 v(i+1, j+1) \\
\leq\ & -r_1 + v(i+1, j+1) + v(i+1, j+1) \\
\leq\ & -r_1 + v(i+2, j) + v(i, j+2) = \bar{T}_1 v(i+1, j) + \bar{T}_1 v(i, j+2),
\end{aligned}$$

where the second inequality follows from the sum of (2.15) and (2.24).

If $a_1 = 0, a_2 = 1$, then

$$\begin{aligned}
& \bar{T}_1 v(i, j+1) + \bar{T}_1 v(i+1, j+1) \\
\leq\ & -r_1 + v(i+1, j+1) + v(i+1, j+1) \\
\leq\ & v(i+1, j) - r_1 + v(i+1, j+2) = \bar{T}_1 v(i+1, j) + \bar{T}_1 v(i, j+2),
\end{aligned}$$

where the second inequality follows from (2.16).

(d) For (2.24), the proof is the same as part (d) of the proof of Lemma 3 with r_2 replaced by r_1 and i, j interchanged.

(e) For (2.25), denote by a the minimizing action in $\bar{T}_1 v(i+1, j)$.

If $a = 0$, then

$$\bar{T}_1 v(i, j+1) \leq v(i, j+1) \leq v(i+1, j) = \bar{T}_1 v(i+1, j).$$

If $a = 1$, then

$$\bar{T}_1 v(i, j+1) \leq -r_1 + v(i+1, j+1) \leq -r_1 + v(i+2, j) = \bar{T}_1 v(i+1, j),$$

where the second inequality follows from (2.25) with i replaced by $i+1$.

□

Lemma 6. *If $v \in \bar{\mathcal{V}}$, then $T_2 v \in \bar{\mathcal{V}}$.*

Proof. T_2 preserving inequalities (2.12) - (2.15) has been proved in Lemma 3. The proof of T_2 preserving (2.24) is the same as part (c) of the proof of Lemma 5 with r_1 replaced by r_2 and i, j interchanged.

For (2.25), denote by a the minimizing action in $T_2 v(i+1, j)$.

If $a = 0$, then

$$T_2 v(i, j+1) \leq v(i, j+1) \leq v(i+1, j) = T_2 v(i+1, j).$$

If $a = 1$, then

$$T_2 v(i, j+1) \leq -r_2 + v(i, j+2) \leq -r_2 + v(i+1, j+1) = T_2 v(i+1, j).$$

□

Lemma 7. *If $v \in \bar{\mathcal{V}}$, then $T_3 v \in \bar{\mathcal{V}}$.*

Proof. T_3 preserving inequalities (2.12) - (2.15) has been proved in Lemma 4.

For (2.24), if $i \geq 1, j \geq 0$, then

$$\begin{aligned} & T_3 v(i+1,j) + T_3 v(i+1,j+1) = v(i,j) + v(i,j+1) \\ \leq \;\; & v(i-1,j+1) + v(i+1,j) = T_3 v(i,j+1) + T_3 v(i+2,j), \end{aligned}$$

where the inequality follows from (2.24) with i replaced by $i-1$.

If $i = 0, j \geq 1$, then

$$\begin{aligned} & T_3 v(1,j) + T_3 v(1,j+1) = v(0,j) + v(0,j+1) \\ \leq \;\; & v(0,j) + v(1,j) = T_3 v(0,j+1) + T_3 v(2,j), \end{aligned}$$

where the inequality follows from (2.25).

If $i = 0, j = 0$, then

$$\begin{aligned} & T_3 v(1,0) + T_3 v(1,1) = v(0,0) + v(0,1) \\ \leq \;\; & v(0,0) + v(1,0) = T_3 v(0,1) + T_3 v(2,0). \end{aligned}$$

For (2.25), if $i \geq 1$, then

$$T_3 v(i,j+1) = v(i-1,j+1) \leq v(i,j) = T_3 v(i+1,j).$$

If $i = 0$, then

$$T_3 v(0,j+1) = v(0,j) = T_3 v(1,j).$$

□

The above lemmas lead to the following theorem.

Theorem 5. *If $h_1 \geq h_2$, the optimal value function $v \in \bar{\mathcal{V}}$.*

Proof. Since $h_1 \geq h_2$, it can be easily shown that $\bar{C}(i,j) \in \bar{\mathcal{V}}$. Lemma 5, 6, 7 show that inequalities (2.12) - (2.15), (2.24), (2.25) are preserved under $\bar{T}_1$, T_2, and T_3. The theorem follows from similar arguments as in the proof of Theorem 3. □

Now we are ready to prove the structural properties of the socially optimal policy.

Theorem 6. *Assume $h_1 \geq h_2$, then the socially optimal policy is characterized by two monotonically decreasing switching curves.*

(1) For each $i \geq 0$, there exists a threshold $L_2^S(i)$, such that a class 2 arrival in state (i, j) is accepted if and only if $j < L_2^S(i)$. Furthermore, $L_2^S(i)$ is monotonically decreasing in i.

(2) For each $j \geq 0$, there exists a threshold $L_1^S(j)$, such that a class 1 arrival in state (i, j) is accepted if and only if $i < L_1^S(j)$. Furthermore, $L_1^S(j)$ is monotonically decreasing in j.

Proof. Define

$$L_1^S(j) = \min\{i : v(i+1, j) - v(i, j) > r_1\},$$

$$L_2^S(i) = \min\{j : v(i, j+1) - v(i, j) > r_2\}.$$

The theorem follows from similar arguments as in the proof of Theorem 4. □

2.5 A Special Case for Social Optimization

We consider the special case where $h_1 = h_2$ under social optimization criterion in this section.

When $h_1 = h_2$, the order of service will not affect the social welfare. So the priority can be ignored and the problem becomes a standard admission control problem with two classes differentiated by different arrival rates and rewards. One can apply the proof in Stidham [42] on both classes and show that the socially optimal policy depends only on the total number of customers in the system and is described by two critical numbers.

We prove this result as a special case of Theorem 6 as follows.

Lemma 8. *If $h_1 = h_2$, then Lemma 5, 6, and 7 hold with (2.25) replaced by*

$$v(i, j+1) = v(i+1, j). \tag{2.27}$$

Proof. We only need to show that (2.27) is preserved under $\bar{T}_1$, T_2, and T_3.

For $\bar{T}_1$, we have

$$\begin{aligned} \bar{T}_1 v(i, j+1) &= \min\{-r_1 + v(i+1, j+1), v(i, j+1)\} \\ &= \min\{-r_1 + v(i+2, j), v(i+1, j)\} = \bar{T}_1 v(i+1, j), \end{aligned}$$

where the second equality follows from the fact that $v(i+1, j+1) = v(i+2, j)$ and $v(i, j+1) = v(i+1, j)$.

T_2 preserving (2.27) can be proved similarly.

For T_3, if $i \geq 1$, then

$$T_3 v(i, j+1) = v(i-1, j+1) = v(i, j) = T_3 v(i+1, j).$$

If $i = 0$, then

$$T_3 v(0, j+1) = v(0, j) = T_3 v(1, j).$$

□

Since $\bar{C}(i, j)$ obviously satisfies (2.27), Lemma 8 implies that Theorem 5 and 6 still hold after replacing (2.25) with (2.27). Thus, we have

Theorem 7. *If $h_1 = h_2$, then there exist constants l_1, l_2 such that*

$$L_1^S(j) = l_1 - j, \tag{2.28}$$

$$L_2^S(i) = l_2 - i, \tag{2.29}$$

where $l_1 \geq l_2$ if and only if $r_1 \geq r_2$.

Proof. Let $l_1 = L_1^S(0)$. In order to prove (2.28), we only need to show that $L_1^S(j+1) = L_1^S(j) - 1$ for any $j \geq 0$.

Let $i' = i + 1$, we have

$$\begin{aligned} L_1^S(j+1) &= \min\{i : v(i+1, j+1) - v(i, j+1) > r_1\} \\ &= \min\{i : v(i+2, j) - v(i+1, j) > r_1\} \\ &= \min\{i' - 1 : v(i'+1, j) - v(i', j) > r_1\} \\ &= \min\{i' : v(i'+1, j) - v(i', j)\} - 1 \\ &= L_1^S(j) - 1, \end{aligned}$$

where the second equality follows from Lemma 8.

(2.29) can be proved similarly by setting $l_2 = L_2^S(0)$.

We have

$$l_1 = L_1^S(0) = \min\{i : v(i+1, 0) - v(i, 0) > r_1\},$$

and

$$\begin{aligned} l_2 = L_2^S(0) &= \min\{j : v(0, j+1) - v(0, j) > r_2\} \\ &= \min\{j : v(j+1, 0) - v(j, 0) > r_2\}, \end{aligned}$$

where the second equality follows from Lemma 8. Therefore, $l_1 \geq l_2$ if and only if

$r_1 \geq r_2$. $\square$

2.6 Comparison and Numerical Results

We compare the optimal policies under different criteria in this section. First, consider the optimal policies for class 1 customers. Under individual optimization criterion, the cost incurred by a class 1 customer is just his own waiting cost (the internal effect). Under class optimization criterion, besides the internal effect, each class 1 customer also causes delay on the class 1 customers joining the system later (the external effect). Under social optimization criterion, the internal effect is the same and the external effect is imposed on all class 2 customers as well as later class 1 customers. Thus, intuitively, accepting a class 1 customer is the most expensive under social optimization and the least expensive under individual optimization. Hence, the number of class 1 customers admitted to the system is the most under individual optimization and the least under social optimization. This intuition is shown to be correct by the following theorem.

Theorem 8. $L_1^S(j) \leq L_1^C \leq L_1^I, \forall j \geq 0$, *where the first inequality holds when* $h_1 \geq h_2$.

Proof. For a $GI/M/1$ single-class queue with convex, nondecreasing holding cost rate, Stidham (1978) proves that more customers are accepted by the individually optimal policy than by the socially optimal policy. As a special case of Stidham's result, we have the second inequality, i.e., $L_1^C \leq L_1^I$. Note that the socially optimal policy in Stidham's model corresponds to the class-optimal policy here.

We prove the first inequality, i.e., $L_1^S(j) \leq L_1^C, \forall j \geq 0$, in the following. Since $L_1^S(j)$ is decreasing in j, we just need to prove $L_1^S(0) \leq L_1^C$. Denote the socially optimal expected total discounted cost by $v^s(i,j)$. When $j = 0$, the optimality equations can be written as

$$\begin{aligned} v^s(i,0) &= h_1 i + \lambda_1 \min\{-r_1 + v^s(i+1,0), v^s(i,0)\} \\ &+ \lambda_2 \min\{-r_2 + v^s(i,1), v^s(i,0)\} + \mu v^s((i-1)^+,0). \end{aligned}$$

Then

$$L_1^S(0) = \min\{i : v^s(i+1,0) - v^s(i,0) > r_1\}. \tag{2.30}$$

Denote the class-optimal expected total discounted cost for controller 1 by $v^c(i)$, the optimality equations can be written as

$$v^c(i) = h_1 i + \lambda_1 \min\{-r_1 + v^c(i+1), v^c(i)\} + \mu v^c((i-1)^+).$$

Then

$$L_1^C = \min\{i : v^c(i+1) - v^c(i) > r_1\}. \tag{2.31}$$

If we can prove

$$v^c(i+1) - v^c(i) \le v^s(i+1,0) - v^s(i,0), \tag{2.32}$$

then the theorem follows.

Apply value iteration. Let $v_0^c(i) = v_0^s(i,0) = 0$, $\forall i$, then (2.32) is satisfied at iteration 0. Suppose (2.32) is true at iteration n, i.e., $v_n^c(i+1) - v_n^c(i) \le v_n^s(i+1,0) - v_n^s(i,0)$. If we can show it is also true at iteration $n+1$ then (2.32) follows by induction and the convergence of value iteration.

$$\begin{aligned} & v_{n+1}^c(i+1) - v_{n+1}^c(i) \\ = \;& h_1 + \lambda_1(\min\{-r_1 + v_n^c(i+2), v_n^c(i+1)\} - \min\{-r_1 + v_n^c(i+1), v_n^c(i)\}) \\ + \;& \mu(v_n^c(i) - v_n^c((i-1)^+)), \end{aligned} \tag{2.33}$$

and

$$
\begin{aligned}
& v^s_{n+1}(i+1,0) - v^s_{n+1}(i,0) \\
= \;& h_1 + \lambda_1(\min\{-r_1 + v^s_n(i+2,0), v^s_n(i+1,0)\} - \min\{-r_1 + v^s_n(i+1,0), v^s_n(i,0)\}) \\
+ \;& \lambda_2(\min\{-r_2 + v^s_n(i+1,1), v^s_n(i+1,0)\} - \min\{-r_2 + v^s_n(i,1), v^s_n(i,0)\}) \\
+ \;& \mu(v^s_n(i,0) - v^s_n((i-1)^+,0)). \qquad (2.34)
\end{aligned}
$$

To simplify notation, let

$$
\begin{aligned}
D^s_1 &= \min\{-r_1 + v^s_n(i+2,0), v^s_n(i+1,0)\} - \min\{-r_1 + v^s_n(i+1,0), v^s_n(i,0)\}, \\
D^s_2 &= \min\{-r_2 + v^s_n(i+1,1), v^s_n(i+1,0)\} - \min\{-r_2 + v^s_n(i,1), v^s_n(i,0)\}, \\
D^s_3 &= v^s_n(i,0) - v^s_n((i-1)^+,0), \\
D^c_1 &= \min\{-r_1 + v^c_n(i+2), v^c_n(i+1)\} - \min\{-r_1 + v^c_n(i+1), v^c_n(i)\}, \\
D^c_3 &= v^c_n(i) - v^c_n((i-1)^+).
\end{aligned}
$$

Compare D^s_1 and D^c_1 first.

Obviously v^c_0 is nondecreasing and convex in i. Following similar approach as in part (c) of the proof of Lemma 3, one can show that if v^c_n is nondecreasing and convex in i, so is v^c_{n+1}. Therefore, if $v^c_n(i+1) - v^c_n(i) > r_1$, then $v^c_n(i+2) - v^c_n(i+1) > r_1$. By induction hypothesis, we also have $v^s_n(i+1,0) - v^s_n(i,0) > r_1$. So

$$D^c_1 = v^c_n(i+1) - v^c_n(i) \leq v^s_n(i+1,0) + v^s_n(i,0) = D^s_1.$$

If $v^c_n(i+1) - v^c_n(i) \leq r_1$ and $v^c_n(i+2) - v^c_n(i+1) > r_1$, then $v^s_n(i+2,0) - v^s_n(i+1,0) >$

r_1. So

$$\begin{aligned} D_1^c &= v_n^c(i+1) - (-r_1 + v_n^c(i+1)) = r_1 \\ &\leq v_n^s(i+1,0) - \min\{-r_1 + v_n^s(i+1,0), v_n^s(i,0)\} = D_1^s. \end{aligned}$$

If $v_n^c(i+1) - v_n^c(i) \leq r_1$, $v_n^c(i+2) - v_n^c(i+1) \leq r_1$, and $v_n^s(i+1,0) - v_n^s(i,0) > r_1$, then $v_n^s(i+2,0) - v_n^s(i+1,0) > r_1$, which follows from (2.26). Thus

$$D_1^c = v_n^c(i+2) - v_n^c(i+1)) \leq r_1 < v_n^s(i+1,0) - v_n^s(i,0) = D_1^s.$$

If $v_n^c(i+1) - v_n^c(i) \leq r_1$, $v_n^c(i+2) - v_n^c(i+1) \leq r_1$, $v_n^s(i+1,0) - v_n^s(i,0) \leq r_1$, and $v_n^s(i+2,0) - v_n^s(i+1,0) > r_1$, then

$$D_1^c = v_n^c(i+2) - v_n^c(i+1)) \leq r_1 = v_n^s(i+1,0) - (-r_1 + v_n^s(i+1,0)) = D_1^s.$$

If $v_n^c(i+1) - v_n^c(i) \leq r_1$, $v_n^c(i+2) - v_n^c(i+1) \leq r_1$, $v_n^s(i+1,0) - v_n^s(i,0) \leq r_1$, and $v_n^s(i+2,0) - v_n^s(i+1,0) \leq r_1$, then

$$\begin{aligned} D_1^c &= v_n^c(i+2) - v_n^c(i+1)) \leq v_n^s(i+2,0) - v_n^s(i+1,0) \\ &= -r_1 + v_n^s(i+2,0) - (-r_1 + v_n^s(i+1,0)) = D_1^s. \end{aligned}$$

Therefore, $D_1^c \leq D_1^s$.

Now consider D_2^s.

If $v_n^s(i+1,1) - v_n^s(i+1,0) \leq r_2$, then $v_n^s(i,1) - v_n^s(i,0) \leq r_2$. So

$$D_2^s = -r_2 + v_n^s(i+1,1) - (-r_2 + v_n^s(i,1)) = v_n^s(i+1,1) - v_n^s(i,1) \geq 0.$$

If $v_n^s(i+1,1) - v_n^s(i+1,0) > r_2$ and $v_n^s(i,1) - v_n^s(i,0) \le r_2$, then

$$D_2^s = v_n^s(i+1,0) - (-r_2 + v_n^s(i,1)) \ge r_2 > 0,$$

which follows from (2.25).

If $v_n^s(i+1,1) - v_n^s(i+1,0) > r_2$ and $v_n^s(i,1) - v_n^s(i,0) > r_2$, then

$$D_2^s = v_n^s(i+1,0) - v_n^s(i,0)) \ge 0.$$

Therefore, $D_2^s \ge 0$.

By induction hypothesis, $D_3^c \le D_3^s$.

Combining the above results, we have

$$v_{n+1}^c(i+1) - v_{n+1}^c(i) \le v_{n+1}^s(i+1,0) - v_{n+1}^s(i,0),$$

thus the theorem follows. □

Now consider the optimal policies for class 2 customers. The external effects of a class 2 customer are the same under class optimization and social optimization. Since the class-optimal policy accepts more class 1 customers than the socially optimal policy, which causes more delay on class 2 customers, the internal effect of a class 2 customer is higher under class optimization than under social optimization. Therefore, intuitively, the class-optimal policy accepts fewer class 2 customers than the socially optimal policy. This intuition is proved to be true by the following theorem.

Theorem 9. *Assume $h_1 \ge h_2$, then $L_2^C(i) \le L_2^S(i), \forall i \ge 0$.*

Proof. We follow similar approach as in the proof of Theorem 8. Denote the socially optimal expected total discounted cost by $v^s(i,j)$. The optimality equations are

$$\begin{aligned} v^s(i,j) &= h_1 i + h_2 j + \lambda_1 \min\{-r_1 + v^s(i+1,j), v^s(i,j)\} \\ &+ \lambda_2 \min\{-r_2 + v^s(i,j+1), v^s(i,j)\} + \mu \begin{cases} v^s(i-1,j), & \text{if } i \geq 1 \\ v^s(0,j-1), & \text{if } i = 0, j \geq 1 \\ v^s(0,0), & \text{if } i = j = 0. \end{cases} \end{aligned}$$

Then

$$L_2^S(i) = \min\{j : v^s(i,j+1) - v^s(i,j) > r_2\}. \tag{2.35}$$

Denote the class-optimal expected total discounted cost for controller 2 by $v^c(i,j)$, the optimality equations are

$$\begin{aligned} v^c(i,j) &= h_2 j + \lambda_1 \begin{cases} v^c(i+1,j), & \text{if } i < L_1^C \\ v^c(i,j), & \text{if } i \geq L_1^C \end{cases} \\ &+ \lambda_2 \min\{-r_2 + v^c(i,j+1), v^c(i,j)\} + \mu \begin{cases} v^c(i-1,j), & \text{if } i \geq 1 \\ v^c(0,j-1), & \text{if } i = 0, j \geq 1 \\ v^c(0,0), & \text{if } i = j = 0. \end{cases} \end{aligned}$$

Then

$$L_2^C(i) = \min\{j : v^c(i,j+1) - v^c(i,j) > r_2\}. \tag{2.36}$$

If we can show

$$v^s(i,j+1) - v^s(i,j) \leq v^c(i,j+1) - v^c(i,j), \forall i, \tag{2.37}$$

then the theorem follows.

Apply value iteration. Let $v_0^c(i,j) = v_0^s(i,j) = v^s(i,j), \forall i,j$, then (2.37) is satisfied at iteration 0. Suppose (2.37) is true at iteration n, i.e., $v_n^s(i,j+1) - v_n^s(i,j) \leq v_n^c(i,j+1) - v_n^c(i,j)$. If we can prove it is also true at iteration $n+1$ then (2.37) follows by induction and the convergence of value iteration.

We have

$$\begin{aligned}
& v^c_{n+1}(i,j+1) - v^c_{n+1}(i,j) \\
= \;& h_2 + \lambda_1 \begin{cases} v^c_n(i+1,j+1) - v^c_n(i+1,j), & \text{if } i < L^C_1 \\ v^c_n(i,j+1) - v^c_n(i,j), & \text{if } i \geq L^C_1 \end{cases} \\
+ \;& \lambda_2(\min\{-r_2 + v^c_n(i,j+2), v^c_n(i,j+1)\} - \min\{-r_2 + v^c_n(i,j+1), v^c_n(i,j)\}) \\
+ \;& \mu \begin{cases} v^c_n(i-1,j+1) - v^c_n(i-1,j), & \text{if } i \geq 1 \\ v^c_n(0,j) - v^c_n(0,j-1), & \text{if } i = 0, j \geq 1 \\ 0, & \text{if } i = j = 0. \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
& v^s_{n+1}(i,j+1) - v^s_{n+1}(i,j) \\
= \;& h_2 + \lambda_1(\min\{-r_1 + v^s_n(i+1,j+1), v^s_n(i,j+1)\} - \min\{-r_1 + v^s_n(i+1,j), v^s_n(i,j)\}) \\
+ \;& \lambda_2(\min\{-r_2 + v^s_n(i,j+2), v^s_n(i,j+1)\} - \min\{-r_2 + v^s_n(i,j+1), v^s_n(i,j)\}) \\
+ \;& \mu \begin{cases} v^s_n(i-1,j+1) - v^s_n(i-1,j), & \text{if } i \geq 1 \\ v^s_n(0,j) - v^s_n(0,j-1), & \text{if } i = 0, j \geq 1 \\ 0, & \text{if } i = j = 0. \end{cases}
\end{aligned}$$

To simplify notation, let

$$\begin{aligned}
D^s_1 &= \min\{-r_1 + v^s_n(i+1,j+1), v^s_n(i,j+1)\} - \min\{-r_1 + v^s_n(i+1,j), v^s_n(i,j)\}, \\
D^s_2 &= \min\{-r_2 + v^s_n(i,j+2), v^s_n(i,j+1)\} - \min\{-r_2 + v^s_n(i,j+1), v^s_n(i,j)\}, \\
D^s_3 &= \begin{cases} v^s_n(i-1,j+1) - v^s_n(i-1,j), & \text{if } i \geq 1 \\ v^s_n(0,j) - v^s_n(0,j-1), & \text{if } i = 0, j \geq 1 \\ 0, & \text{if } i = j = 0, \end{cases}
\end{aligned}$$

$$D_1^c = \begin{cases} v_n^c(i+1,j+1) - v_n^c(i+1,j), & \text{if } i < L_1^C \\ v_n^c(i,j+1) - v_n^c(i,j), & \text{if } i \geq L_1^C, \end{cases}$$

$$D_2^c = \min\{-r_2 + v_n^c(i,j+2), v_n^c(i,j+1)\} - \min\{-r_2 + v_n^c(i,j+1), v_n^c(i,j)\},$$

$$D_3^c = \begin{cases} v_n^c(i-1,j+1) - v_n^c(i-1,j), & \text{if } i \geq 1 \\ v_n^c(0,j) - v_n^c(0,j-1), & \text{if } i = 0, j \geq 1 \\ 0, & \text{if } i = j = 0. \end{cases}$$

Compare D_1^s and D_1^c first.

If $v_n^s(i+1,j) - v_n^s(i,j) > r_1$, then $v_n^s(i+1,j+1) - v_n^s(i,j+1) > r_1$, which follows from (2.14). So

$$D_1^s = v_n^s(i,j+1) - v_n^s(i,j) \leq v_n^c(i,j+1) - v_n^c(i,j) \leq D_1^c.$$

Since $L_1^S(j) \leq L_1^C, \forall j$, class 1 arrivals in state (i,j) with $i \geq L_1^C$ are always rejected by the socially optimal policy, i.e., $v^s(i+1,j) - v^s(i,j) > r_1$, $\forall i \geq L_1^C$. Since $v_0^s(i,j) = v^s(i,j)$, we have $v_k^s(i,j) = v^s(i,j)$, $\forall k \geq 0$. Hence, $v_k^s(i+1,j) - v_k^s(i,j) > r_1$, $\forall k \geq 0, i \geq L_1^C$.

If $v_n^s(i+1,j) - v_n^s(i,j) \leq r_1$ and $v_n^s(i+1,j+1) - v_n^s(i,j+1) > r_1$, the above observation yields $i < L_1^C$. So

$$\begin{aligned} D_1^s &= v_n^s(i,j+1) - (-r_1 + v_n^s(i+1,j)) \\ &\leq v_n^s(i,j+1) + (v_n^s(i+1,j+1) - v_n^s(i,j+1)) - v_n^s(i+1,j) \\ &= v_n^s(i+1,j+1) - v_n^s(i+1,j) \leq v_n^c(i+1,j+1) - v_n^c(i+1,j) = D_1^c. \end{aligned}$$

If $v_n^s(i+1,j) - v_n^s(i,j) \leq r_1$, $v_n^s(i+1,j+1) - v_n^s(i,j+1) \leq r_1$, then $i < L_1^C$. So

$$\begin{aligned} D_1^s &= -r_1 + v_n^s(i+1,j+1) - (-r_1 + v_n^s(i+1,j)) \\ &= v_n^s(i+1,j+1) - v_n^s(i+1,j) \leq v_n^c(i+1,j+1) - v_n^c(i+1,j) = D_1^c. \end{aligned}$$

Therefore, $D_1^s \leq D_1^c$.

Now consider D_2^s and D_2^c.

If $v_n^s(i,j+1) - v_n^s(i,j) > r_2$, then $v_n^s(i,j+2) - v_n^s(i,j+1) > r_2$, which follows from (2.16). By induction hypothesis, we have $v_n^c(i,j+1) - v_n^c(i,j) > r_2$ and $v_n^c(i,j+2) - v_n^c(i,j+1) > r_2$. So

$$D_2^s = v_n^s(i,j+1) - v_n^s(i,j) \leq v_n^c(i,j+1) - v_n^c(i,j) = D_2^c.$$

If $v_n^s(i,j+1) - v_n^s(i,j) \leq r_2$ and $v_n^s(i,j+2) - v_n^s(i,j+1) > r_2$. Then $v_n^c(i,j+2) - v_n^c(i,j+1) > r_2$. So

$$\begin{aligned} D_2^s &= v_n^s(i,j+1) - (-r_2 + v_n^s(i,j+1)) = r_2 \\ &\leq v_n^c(i,j+1) - \min\{-r_2 + v_n^c(i,j+1), v_n^c(i,j)\} = D_2^c. \end{aligned}$$

If $v_n^s(i,j+1) - v_n^s(i,j) \leq r_2$, $v_n^s(i,j+2) - v_n^s(i,j+1) \leq r_2$, $v_n^c(i,j+1) - v_n^c(i,j) > r_2$, then $v_n^c(i,j+2) - v_n^c(i,j+1) > r_2$, which follows from (2.16). So

$$\begin{aligned} D_2^s &= v_n^s(i,j+2) - v_n^s(i,j+1)) \leq r_2 \\ &< v_n^c(i,j+1) - v_n^c(i,j) = D_2^c. \end{aligned}$$

If $v_n^s(i,j+1) - v_n^s(i,j) \leq r_2$, $v_n^s(i,j+2) - v_n^s(i,j+1) \leq r_2$, $v_n^c(i,j+1) - v_n^c(i,j) \leq r_2$, and $v_n^c(i,j+2) - v_n^c(i,j+1) > r_2$, then

$$\begin{aligned} D_2^s &= v_n^s(i,j+2) - v_n^s(i,j+1)) \leq r_2 \\ &= v_n^c(i,j+1) - (-r_2 + v_n^c(i,j+1)) = D_2^c. \end{aligned}$$

If $v_n^s(i,j+1) - v_n^s(i,j) \leq r_2$, $v_n^s(i,j+2) - v_n^s(i,j+1) \leq r_2$, $v_n^c(i,j+1) - v_n^c(i,j) \leq r_2$, and $v_n^c(i,j+2) - v_n^c(i,j+1) \leq r_2$, then

$$D_2^s = -r_2+v_n^s(i,j+2)-(-r_2+v_n^s(i,j+1)) \leq -r_2+v_n^c(i,j+2)-(-r_2+v_n^c(i,j+1)) = D_2^c.$$

Therefore, $D_2^s \leq D_2^c$.

By induction hypothesis, $D_3^s \leq D_3^c$.

Combining the above results, we have

$$v_{n+1}^s(i,j+1) - v_{n+1}^s(i,j) \leq v_{n+1}^c(i,j+1) - v_{n+1}^c(i,j),$$

thus the theorem follows. □

It is worth noting that the comparisons between class-optimal and socially optimal policies give opposite results for class 1 and class 2. This contrast has the following interesting socioeconomic connotation. Suppose the whole society can be divided into two classes, influentials and grass roots. If we define "better" as "more people get served", then the influentials will prefer to optimize things within their own class, while the grass roots will be better off if the society is centrally controlled by a decision maker who can take their benefits into consideration. Seen in this fashion, the result makes intuitive sense.

Now compare the individually optimal policy with the other two optimal policies. Under individual optimization, a class 2 customer has no external effect, but it has more internal effect than under class or social optimization, since the individually optimal policy accepts the most class 1 customers. So the comparison results between $L_2^I(i)$ and $L_2^C(i)$ and between $L_2^I(i)$ and $L_2^C(i)$ depend on which effect is dominant.

We demonstrate the above results by numerical examples below. The numerical examples are computed by using standard value iteration algorithm. We approximate the infinite state space by assuming that no customers arrive when the total number of customers in the system reaches an upper bound B, which is much larger than the expected queue length. Thus the state space is $S = \{(i,j) : 0 \leq i,j \leq B\}$. The stopping criterion is $\max\{|v_{n+1}(i,j) - v_n(i,j)| : (i,j) \in S\} \leq 10^{-5}$, where $v_n(i,j)$ is the value function at the n^{th} iteration.

Figure 2.1 illustrates the optimal policies for class 1 customers with parameters $\alpha = 0.05, \mu = 0.5, \lambda_1 = 0.44, \lambda_2 = 0.01, h_1 = 20, h_2 = 10, r_1 = 200, r_2 = 190$. Figure 2.2 - 2.6 illustrate the optimal policies for class 2 customers under different arrival rates. Figure 2.2 uses the same parameters as used in Figure 2.1 and shows that $L_2^I(i) \leq L_2^C(i) \leq L_2^S(i), \forall i$. Keeping the other parameters the same, Figure 2.3 uses $\lambda_1 = 0.39, \lambda_2 = 0.06$, and shows that $L_2^C(i) \leq L_2^I(i) \leq L_2^S(i), \forall i$. Figure 2.4 uses $\lambda_1 = 0.27, \lambda_2 = 0.18$, and shows that $L_2^C(i) \leq L_2^S(i) \leq L_2^I(i), \forall i$. Figure 2.5 uses $\lambda_1 = 0.41, \lambda_2 = 0.04$, and shows that $L_2^I(i) \geq L_2^C(i)$ for $i \leq 4, L_2^I(i) \leq L_2^C(i)$ for $i \geq 5$. Figure 2.6 uses $\lambda_1 = 0.32, \lambda_2 = 0.13$, and shows that $L_2^I(i) \leq L_2^S(i)$ for $i \leq 7, L_2^I(i) \geq L_2^S(i)$ for $i \geq 8$. We only change the arrival rates in the above examples. However, other numerical examples show that changing other parameters may also affect the relative position of $L_2^I(i)$. Thus the relationship between the individually optimal policy and either of the other two optimal policies can be arbitrary depending on the parameters.

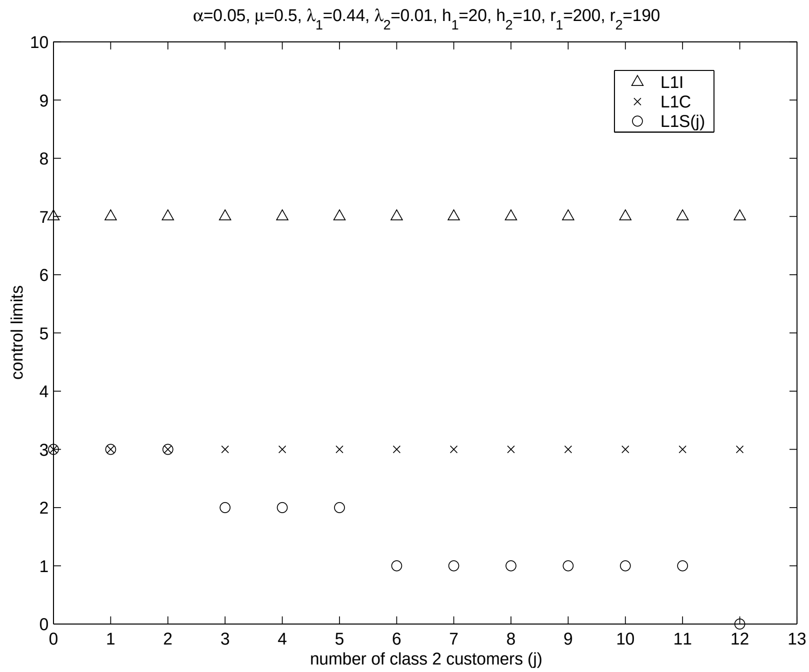

Figure 2.1: Class 1 switching curves: $L_1^S(j) \leq L_1^C \leq L_1^I$

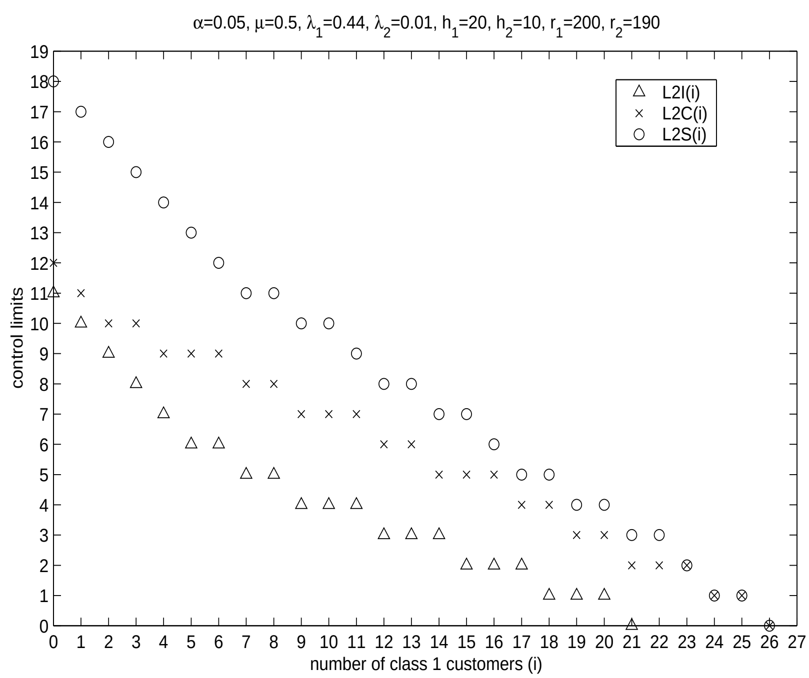

Figure 2.2: Class 2 switching curves: $L_2^I(i) \leq L_2^C(i) \leq L_2^S(i)$

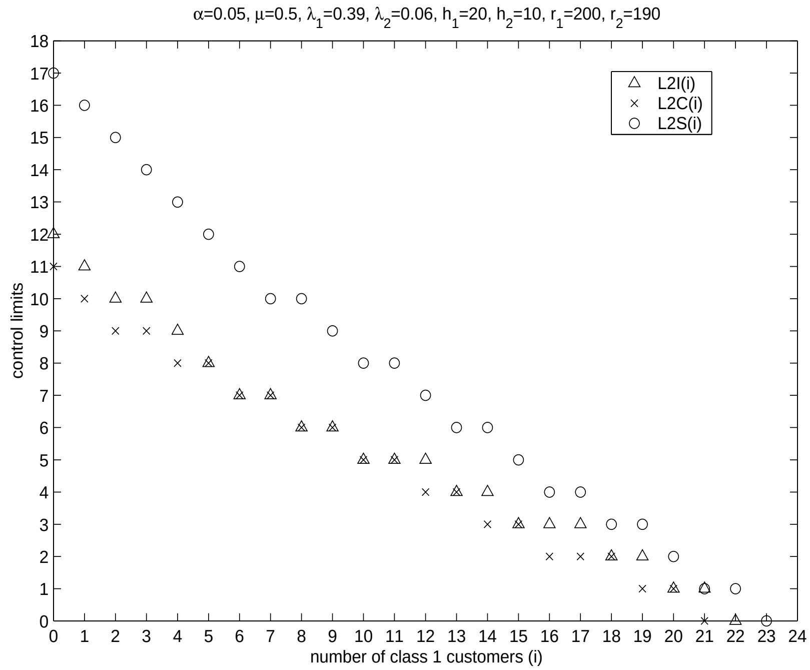

Figure 2.3: Class 2 switching curves: $L_2^C(i) \leq L_2^I(i) \leq L_2^S(i)$

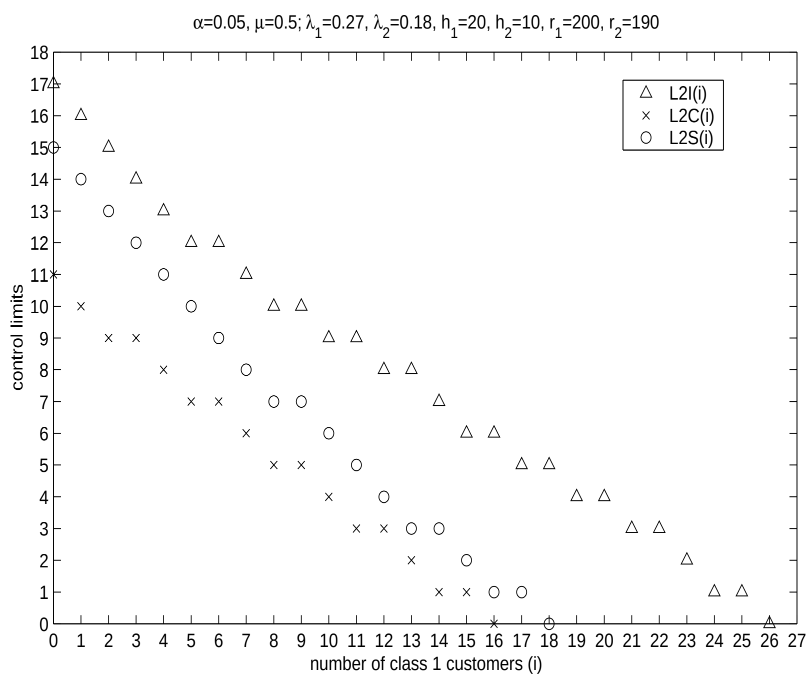

Figure 2.4: Class 2 switching curves: $L_2^C(i) \leq L_2^S(i) \leq L_2^I(i)$

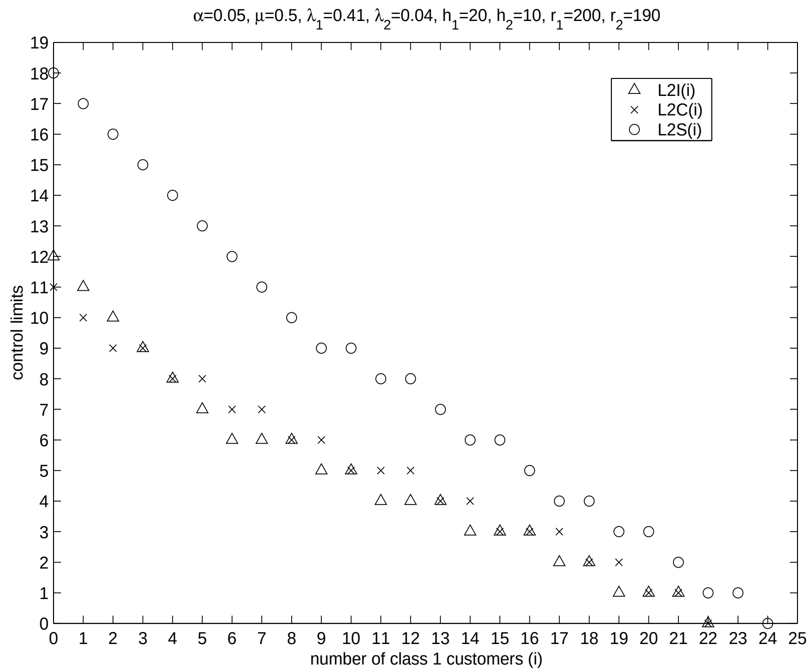

Figure 2.5: Class 2 switching curves: $L_2^I(i) \geq L_2^C(i)$ for $i \leq 4, L_2^I(i) \leq L_2^C(i)$ for $i \geq 5$

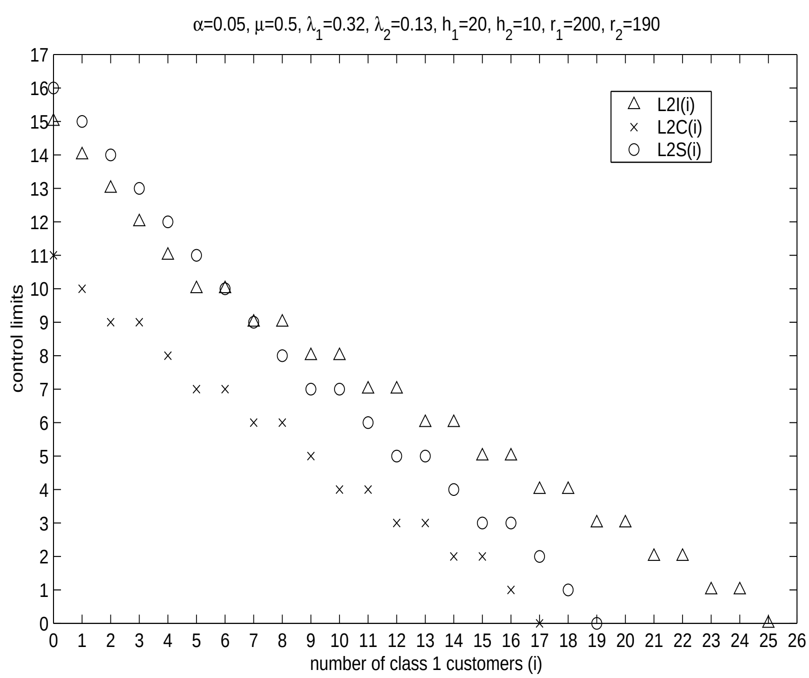

Figure 2.6: Class 2 switching curves: $L_2^I(i) \leq L_2^S(i)$ for $i \leq 7, L_2^I(i) \geq L_2^S(i)$ for $i \geq 8$

Chapter 3

Admission Control: Sample Path Approach

3.1 Problem Description

In this chapter, we study the multi-priority admission control problem as defined in Chapter 2 with the following two differences: (i) Rewards are generated at the time of service completion instead of the time of joining the repair queue. This shift of reward time changes the nature of the problem in some critical ways, e.g. the optimal value function is no longer non-decreasing in the number of customers of each type in initial state, and the cases where every customer is accepted do not exist anymore. (ii) We prove the structural results using sample path analysis (specifically, the coupling method) (Lindvall [26], Wu et al. [47]) instead of standard value iteration method as used in Chapter 2. The sample path approach provides more concise proofs.

We analyze the optimal control policy under the 3 criteria proposed in Chapter 2, i.e., individual optimization, class optimization, and social optimization. We also compare different policies numerically.

3.2 Individual Optimization

Following the same approach as the proof of Theorem 1, we can derive the following results for individually optimal policies.

Theorem 10. *Under the individual optimization criterion, an arriving class 1 customer who sees the system in state (i, j) joins the queue if and only if $i < L_1^I$, where*

$$L_1^I = \lfloor \log \frac{h_1}{h_1 + \alpha r_1} / \log \frac{\mu}{\mu + \alpha} \rfloor. \tag{3.1}$$

An arriving class 2 customer who sees the system in state (i, j) joins the queue if and only if $j < L_2^I(i)$, where

$$L_2^I(i) = \begin{cases} \lfloor \log \frac{h_2}{(h_2+\alpha r_2)\phi_i(\alpha)} / \log \beta \rfloor, & \text{if } \; i \leq L_1^I \\ \lfloor (\log \frac{h_2}{(h_2+\alpha r_2)\phi_{L_1^I}(\alpha)} + (i - L_1^I)(\log \frac{\mu+\alpha}{\mu})) / \log \beta \rfloor, & \text{if } \; i > L_1^I \end{cases} \tag{3.2}$$

where $\phi_i(\alpha)$ is the LST of the busy period initiated by i class 1 customers and $\beta = \frac{\mu}{\alpha+\mu+\lambda_1(1-\phi_1(\alpha))}$. $\lfloor x \rfloor$ is the largest integer less than or equal to x. Furthermore, $L_2^I(i)$ is decreasing in i.

Note that shifting the reward time (from the moment a customer joins the queue to the moment a customer finishes service) not only changes the form of the threshold functions but also eliminates the cases where everyone is accepted.

3.3 Social Optimization

We consider socially optimal policies in this section. The objective of a socially optimal policy is to minimize the expected total discounted net cost generated by all customers. Let $v(i, j)$ be the expected total discounted net cost generated by a socially optimal policy over an infinite horizon starting from state (i, j). Following

Lippman [27], we uniformize the process by defining the uniform rate $\Lambda = \lambda_1 + \lambda_2 + \mu$. Rescaling time so that $\Lambda + \alpha = 1$, we have the following optimality equations

$$
\begin{aligned}
v(i,j) &= h_1 i + h_2 j + \lambda_1 \min\{v(i,j), v(i+1,j)\} \\
&\quad + \lambda_2 \min\{v(i,j), v(i,j+1)\} \\
&\quad + \mu \begin{cases} v(i-1,j) - r_1, & \text{if } i \geq 1 \\ v(0,j-1) - r_2, & \text{if } i = 0, j \geq 1 \\ v(0,0), & \text{if } i = 0, j = 0. \end{cases}
\end{aligned} \tag{3.3}
$$

Lemma 9. $v(0,1) - v(0,0) + r_2 \geq 0$.

Proof. Define two processes on the same probability space so that they see the same arrivals and potential services. Process 1 starts in state $(0,1)$ and follows optimal policy. Process 2 starts in state $(0,0)$ and follows policy ϕ which is described below. Let τ be the first time Process 1 reaches state $(0,0)$. Let Process 2 take the same action as Process 1 upon each arrival until time τ, then follow the optimal policy afterwards. If a new class 2 customer is accepted while Process 1 is serving the last class 2 customer, we resample the remaining service time of the class 2 customer currently under service in Process 1 so that he finishes service at the same time as the new class 2 customer in Process 2. (This resampling argument can be applied to similar situations in the rest of this paper.) Therefore, Process 1 and 2 have identical customers except for one extra class 2 customer in Process 1 until time τ. Two processes become identical from then on. Thus,

$$
\begin{aligned}
v(0,1) - v(0,0) &\geq v(0,1) - v^{\phi}(0,0) \\
&= E\int_0^{\tau} e^{-\alpha t} h_2 dt + Ee^{-\alpha\tau}(-r_2 + v(0,0) - v(0,0)) \\
&\geq -r_2 Ee^{-\alpha\tau} \geq -r_2.
\end{aligned}
$$

$\square$

Lemma 10. *v is supermodular, i.e.,*

$$v(i+1,j+1) - v(i+1,j) - v(i,j+1) + v(i,j) \geq 0. \tag{3.4}$$

Proof. Fix i and j. Define four processes on the same probability space so that they see the same arrivals and potential services. Process 1 and 4 follow optimal policies and start in states $(i+1, j+1)$ and (i,j), respectively. Process 2 and 3 start in states $(i+1,j)$ and $(i,j+1)$, respectively, and use policies ϕ_2 and ϕ_3 which are described below. Denote the state of Process k at time t by (X_t^k, Y_t^k), $k = 1, 2, 3, 4$.

Let τ_1 be the first time Process 2 and 3 have 0 customers entirely. Note that if Process 2 and 3 take the same action upon each arrival they will reach state (0,0) at the same time, since service rates are the same for both classes. Let τ_2 be the first time Process 1 and 4 take different actions. Define $\tau = \min\{\tau_1, \tau_2\}$. Let Process 2 and 3 take the same action as Process 1 and 4 until time τ, then follow the optimal policy afterwards. Thus

$$\begin{aligned}
& v(i+1,j+1) - v(i+1,j) - v(i,j+1) + v(i,j) \\
\geq\ & v(i+1,j+1) - v^{\phi_2}(i+1,j) - v^{\phi_3}(i,j+1) + v(i,j) \\
=\ & E\int_0^\tau e^{-\alpha t}[h(X_t^4+1, Y_t^4+1) - h(X_t^4+1, Y_t^4) - h(X_t^4, Y_t^4+1) + h(X_t^4, Y_t^4)]dt \\
& + Ee^{-\alpha\tau}(-R_1 + R_2 + R_3 - R_4) \\
& + Ee^{-\alpha\tau}(v(X_\tau^1, Y_\tau^1) - v(X_\tau^2, Y_\tau^2) - v(X_\tau^3, Y_\tau^3) + v(X_\tau^4, Y_\tau^4)),
\end{aligned}$$

where R_i is the potential reward generated in Process i at time τ. It can be easily seen that the first term is 0 because of the linear holding cost rate.

To simplify notation, define

$$D = v(i+1,j+1) - v(i+1,j) - v(i,j+1) + v(i,j) \tag{3.5}$$

$$A = -R_1 + R_2 + R_3 - R_4 \tag{3.6}$$

$$B = v(X_\tau^1, Y_\tau^1) - v(X_\tau^2, Y_\tau^2) - v(X_\tau^3, Y_\tau^3) + v(X_\tau^4, Y_\tau^4). \tag{3.7}$$

Case 1: $\tau = \tau_1$. Then, at τ, the four processes are in states $(0,1)$, $(0,0)$, $(0,0)$, and $(0,0)$, respectively. The two distinct paths by which this state is reached are: (i) $\{(1,2)\ (1,1)\ (0,2)\ (0,1)\} \to \{(0,2)\ (0,1)\ (0,1)\ (0,0)\} \to \{(0,1)\ (0,0)\ (0,0)\ (0,0)\}$; (ii) $\{(2,1)\ (2,0)\ (1,1)\ (1,0)\} \to \{(1,1)\ (1,0)\ (0,1)\ (0,0)\} \to \{(0,1)\ (0,0)\ (0,0)\ (0,0)\}$. In the former case, $R_1 = R_2 = R_3 = r_2$, and $R_4 = 0$. In the latter case, $R_1 = R_2 = r_1$, $R_3 = r_2$, and $R_4 = 0$. In both cases, we have

$$D \geq Ee^{-\alpha\tau}(r_2 + v(0,1) - v(0,0)) \geq 0,$$

where the last inequality follows from Lemma 9.

Case 2: $\tau = \tau_2$. Then $A = 0$. We have the following possibilities.

Case 2.1: A class 1 arrival is accepted by Process 1 and rejected by Process 4. Let Process 2 accept the arrival and Process 3 reject it. Then after this event the states in four processes are $(X_\tau^4 + 2, Y_\tau^4 + 1)$, $(X_\tau^4 + 2, Y_\tau^4)$, $(X_\tau^4, Y_\tau^4 + 1)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 + 1, Y_\tau^4 + 1) + v(X_\tau^4 + 1, Y_\tau^4)$, we have

$$\begin{aligned} B &= v(X_\tau^4 + 2, Y_\tau^4 + 1) - v(X_\tau^4 + 1, Y_\tau^4 + 1) - v(X_\tau^4 + 2, Y_\tau^4) + v(X_\tau^4 + 1, Y_\tau^4) \\ &\quad + v(X_\tau^4 + 1, Y_\tau^4 + 1) - v(X_\tau^4, Y_\tau^4 + 1) - v(X_\tau^4 + 1, Y_\tau^4) + v(X_\tau^4, Y_\tau^4). \end{aligned}$$

Note that the first four terms and the second four terms above are inequality (3.4) evaluated at $(X_\tau^4 + 1, Y_\tau^4)$ and (X_τ^4, Y_τ^4), respectively. Thus the above argument can be repeated until either Case 1 or Case 2.2 or Case 2.4 happens.

Case 2.2: A class 1 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 reject the arrival and Process 3 accept it. Then after this event the states in four processes are (X_τ^4+1, Y_τ^4+1), (X_τ^4+1, Y_τ^4), (X_τ^4+1, Y_τ^4+1), and (X_τ^4+1, Y_τ^4), respectively. Note that Process 1 and 3 couple, so do Process 2 and 4. Therefore $B = 0$ and (3.4) holds.

Case 2.3: A class 2 arrival is accepted by Process 1 and rejected by Process 4. Let Process 2 reject the arrival and Process 3 accept it. Then after this event the states in four processes are $(X_\tau^4 + 1, Y_\tau^4 + 2)$, $(X_\tau^4 + 1, Y_\tau^4)$, $(X_\tau^4, Y_\tau^4 + 2)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 + 1, Y_\tau^4 + 1) + v(X_\tau^4, Y_\tau^4 + 1)$, we have

$$\begin{aligned} B &= v(X_\tau^4 + 1, Y_\tau^4 + 2) - v(X_\tau^4 + 1, Y_\tau^4 + 1) - v(X_\tau^4, Y_\tau^4 + 2) + v(X_\tau^4, Y_\tau^4 + 1) \\ &\quad + v(X_\tau^4 + 1, Y_\tau^4 + 1) - v(X_\tau^4 + 1, Y_\tau^4) - v(X_\tau^4, Y_\tau^4 + 1) + v(X_\tau^4, Y_\tau^4). \end{aligned}$$

Note that the first four terms and the second four terms are inequality (3.4) evaluated at $(X_\tau^4, Y_\tau^4 + 1)$ and (X_τ^4, Y_τ^4), respectively. Thus the above argument can be repeated until either Case 1 or Case 2.2 or Case 2.4 happens.

Case 2.4: A class 2 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 accept the arrival and Process 3 reject it. Then after this event the states in four processes are $(X_\tau^4 + 1, Y_\tau^4 + 1)$, $(X_\tau^4 + 1, Y_\tau^4 + 1)$, $(X_\tau^4, Y_\tau^4 + 1)$, and $(X_\tau^4, Y_\tau^4 + 1)$, respectively. Note that Process 1 and 2 couple, so do Process 3 and 4. Therefore $B = 0$ and (3.4) holds. $\square$

Lemma 11. *$v(i, j)$ is a unimodal function in i, i.e., if $v(i + 1, j) - v(i, j) \geq 0$, then $v(i + 2, j) - v(i + 1, j) \geq 0$.*

Proof. Define two processes on the same probability space so that they see the same arrivals and potential services. Process 1 follows the optimal policy and starts in state $(i + 2, j)$. Process 2 starts in state $(i + 1, j)$ and follows policy ϕ that is described below.

Let τ be the first time Process 1 has $i + 1$ class 1 customers. Process 2 takes the same action as Process 1 upon arrivals until τ then follow the optimal policy afterwards. Thus, at time τ Process 2 has i class 1 customers and the same number of class 2 customers, say j', as in Process 1. We have $j' \geq j$, since no class 2 customers have started service yet. Hence

$$\begin{aligned} v(i+2,j) - v(i+1,j) &\geq v(i+2,j) - v^{\phi}(i+1,j) \\ &= E\int_0^{\tau} e^{-\alpha t} h_1 dt + Ee^{-\alpha\tau}(v(i+1,j') - v(i,j')), \end{aligned}$$

where $j' \geq j$. From supermodularity, we have

$$v(i+1,j') - v(i,j') \geq v(i+1,j) - v(i,j) \geq 0.$$

Therefore $v(i+2,j) - v(i+1,j) \geq 0$. □

Theorem 11. *The socially optimal policy for admitting class 1 customers is characterized by a monotonically decreasing switching curve, i.e., for each $j \geq 0$, there exists a threshold $L_1^s(j)$, such that a class 1 arrival in state (i,j) is accepted if and only if $i < L_1^s(j)$. Furthermore, $L_1^s(j)$ is monotonically decreasing in j.*

Proof. We follow the convention that a customer is accepted when accepting or rejecting that customer makes no difference in terms of cost. Then a class 1 arrival in state (i,j) is accepted if and only if $v(i+1,j) \leq v(i,j)$. For any fixed j, let

$$L_1^s(j) = \min\{i : v(i+1,j) > v(i,j)\}.$$

Using Lemma 11, one can easily show that a class 1 arrival is accepted if and only if $i < L_1^s(j)$.

For $j_1 \leq j_2$, we have $v(i+1,j_2) - v(i,j_2) \geq v(i+1,j_1) - v(i,j_1)$, which follows from supermodularity. By definition of $L_1^s(j_1)$, we have $v(L_1^s(j_1)+1,j_1) > v(L_1(j_1),j_1)$, so $v(L_1^s(j_1)+1,j_2) > v(L_1^s(j_1),j_2)$. By definition of $L_1^s(j_2)$, we have $L_1^s(j_1) \geq L_1^s(j_2)$. Thus, $L_1^s(j)$ is decreasing in j. □

Lemma 12. *If $h_1 \geq h_2$ and $r_1 \geq r_2$, then v is diagonally dominant in both i and j, i.e.,*

$$v(i,j+2) - v(i,j+1) - v(i+1,j+1) + v(i+1,j) \geq 0, \tag{3.8}$$

$$v(i,j+1) - v(i+1,j) - v(i+1,j+1) + v(i+2,j) \geq 0. \tag{3.9}$$

Proof. (a). Consider (3.8) first.

Define four processes on the same probability space so that they see the same arrivals and potential services. Process 1 and 4 follow optimal policies and start in state $(i,j+2)$ and $(i+1,j)$, respectively. Process 2 and 3 start in state $(i,j+1)$ and $(i+1,j+1)$, respectively, and use policies ϕ_2 and ϕ_3 which are described below. Denote the state of Process k at time t by (X_t^k, Y_t^k), $k = 1,2,3,4$.

Let τ_1 be the first time Process 3 and 4 have 0 class 1 customers. Since service rates are the same for both classes, Process 1 and 2 finish serving the first class 2 customer at τ_1. Let τ_2 be the first time Process 1 and 4 take different actions. Define $\tau = \min\{\tau_1, \tau_2\}$. Let Process 2 and 3 take the same action as Process 1 and 4 upon each arrival until time τ, then follow the optimal policy afterwards. Thus

$$\begin{aligned}
& v(i,j+2) - v(i,j+1) - v(i+1,j+1) + v(i+1,j) \\
\geq \;& v(i,j+2) - v^{\phi_2}(i,j+1) - v^{\phi_3}(i+1,j+1) + v(i+1,j) \\
= \;& E\int_0^\tau e^{-\alpha t}[h(X_t^4-1, Y_t^4+2) - h(X_t^4-1, Y_t^4+1) - h(X_t^4, Y_t^4+1) + h(X_t^4, Y_t^4)]dt \\
& + Ee^{-\alpha\tau}(-R_1 + R_2 + R_3 - R_4) \\
& + Ee^{-\alpha\tau}(v(X_\tau^1, Y_\tau^1) - v(X_\tau^2, Y_\tau^2) - v(X_\tau^3, Y_\tau^3) + v(X_\tau^4, Y_\tau^4)),
\end{aligned}$$

where R_i is the potential reward generated in Process i at time τ. It can be easily seen that the first term is 0 because of the linear holding cost rate.

Define A, B as in (3.6), (3.7).

Case a.1: $\tau = \tau_1$. Then the states in four processes at time τ are $(0, Y_\tau^4 + 1)$, $(0, Y_\tau^4)$,

$(0, Y_\tau^4+1)$, and $(0, Y_\tau^4)$, respectively. Note that Process 1 and 3 couple, so do Process 2 and 4. Therefore $B = 0$. Also, $R_1 = R_2 = r_2$ and $R_3 = R_4 = r_1$, so $A = 0$. Thus (3.8) holds.

Case a.2: $\tau = \tau_2$. Then $A = 0$. We have the following possibilities.

Case a.2.1: A class 1 arrival is accepted by Process 1 and rejected by Process 4. Let Process 2 accept the arrival and Process 3 reject it. Then the states in four processes at time τ are $(X_\tau^4, Y_\tau^4 + 2)$, $(X_\tau^4, Y_\tau^4 + 1)$, $(X_\tau^4, Y_\tau^4 + 1)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 + 1, Y_\tau^4 + 1) + v(X_\tau^4 + 1, Y_\tau^4)$, we have

$$\begin{aligned} B &= v(X_\tau^4, Y_\tau^4 + 2) - v(X_\tau^4 + 1, Y_\tau^4 + 1) - v(X_\tau^4, Y_\tau^4 + 1) + v(X_\tau^4 + 1, Y_\tau^4) \\ &\quad + v(X_\tau^4 + 1, Y_\tau^4 + 1) - v(X_\tau^4, Y_\tau^4 + 1) - v(X_\tau^4 + 1, Y_\tau^4) + v(X_\tau^4, Y_\tau^4). \end{aligned}$$

Note that the first four terms are inequality (3.8) evaluated at (X_τ^4, Y_τ^4), so the above argument can be repeated until Case a.1 or Case a.2.4 happens. The second four terms are inequality (3.4) evaluated at (X_τ^4, Y_τ^4), which is non-negative by Lemma 10.

Case a.2.2: A class 1 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 accept the arrival and Process 3 reject it. Then the states in four processes at time τ are $(X_\tau^4 - 2, Y_\tau^4 + 2)$, $(X_\tau^4 - 1, Y_\tau^4 + 1)$, $(X_\tau^4 - 1, Y_\tau^4 + 1)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 - 2, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4)$, we have

$$\begin{aligned} B &= v(X_\tau^4 - 2, Y_\tau^4 + 2) - v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4) \\ &\quad + v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4) + v(X_\tau^4, Y_\tau^4). \end{aligned}$$

Note that the first four terms are inequality (3.8) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, so the above argument can be repeated until Case a.1 or Case a.2.4 happens. The second four terms are inequality (3.9) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, so the argument in part (b) can be repeated until Case b.1 or Case b.2.1 happens.

Case a.2.3: A class 2 arrival is accepted by Process 1 and rejected by Process 4. Let

Process 2 accept the arrival and Process 3 reject it. Then the states in four processes at time τ are (X_τ^4-1, Y_τ^4+3), (X_τ^4-1, Y_τ^4+2), (X_τ^4, Y_τ^4+1), and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4-1, Y_\tau^4+2)+v(X_\tau^4, Y_\tau^4+2)+v(X_\tau^4, Y_\tau^4+1)$, we have

$$\begin{aligned} B &= v(X_\tau^4-1, Y_\tau^4+3)-v(X_\tau^4-1, Y_\tau^4+2)-v(X_\tau^4, Y_\tau^4+2)+v(X_\tau^4, Y_\tau^4+1) \\ &\quad +v(X_\tau^4-1, Y_\tau^4+2)-v(X_\tau^4-1, Y_\tau^4+1)-v(X_\tau^4, Y_\tau^4+1)+v(X_\tau^4, Y_\tau^4) \\ &\quad +v(X_\tau^4-1, Y_\tau^4+1)-v(X_\tau^4-1, Y_\tau^4+2)-v(X_\tau^4, Y_\tau^4+1)+v(X_\tau^4, Y_\tau^4+2). \end{aligned}$$

Note that the first four terms and the second four terms are inequality (3.8) evaluated at (X_τ^4-1, Y_τ^4+1) and (X_τ^4-1, Y_τ^4), respectively. So the above argument can be repeated until Case a.1 or Case a.2.4 happens. The last four terms are inequality (3.4) evaluated at (X_τ^4-1, Y_τ^4+1), which is non-negative by Lemma 10.

Case a.2.4: A class 2 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 accept the arrival and Process 3 reject it. Then the states in four processes at time τ are (X_τ^4-1, Y_τ^4+1), (X_τ^4-1, Y_τ^4+1), (X_τ^4, Y_τ^4), (X_τ^4, Y_τ^4), respectively. Note that Process 1 and 2 couple, so do Process 3 and 4. Therefore $B=0$ and hence (3.8) holds.

(b). Consider (3.9) next.

Define four processes on the same probability space so that they see the same arrivals and potential services. Process 1 and 4 follow optimal policies and start in state $(i, j+1)$ and $(i+2, j)$, respectively. Process 2 and 3 start in state $(i+1, j)$ and $(i+1, j+1)$, respectively, and use policies ϕ_2 and ϕ_3 which are described below. Denote the state of Process k at time t by (X_t^k, Y_t^k), $k=1,2,3,4$.

Let β be the first time Process 2 and 3 have 0 class 1 customers. Let τ_1 be the first time Process 4 has 0 class 1 customers. Since service rates are the same for both classes, Process 1 finishes serving the first class 2 customer at β and the second class 2 customer (if any) at τ_1. Process 2 and 3 finish serving the first class 2 customer (if any) at τ_1. So between β and τ_1, Process 1 and 2 have identical customers, and Process 3 has one more class 2 customer but one less class 1 customer than Process 4. While Process 4 is serving the last class 1 customer, the servers in Process 1 and 2 are either serving class 2 customers or idle. In the former case, the rewards generated in four processes at τ_1 are respectively r_2, r_2, r_2, and r_1. In the latter case, the rewards are respectively $0, 0, r_2$, and r_1. Let τ_2 be the first time Process 1 and 4 take different actions. Define $\tau = \min\{\tau_1, \tau_2\}$. Let Process 2 and 3 take the same action as Process 1 and 4 upon each arrival until time τ, then follow the optimal policy afterwards.

Case b.1: $\tau = \tau_1$. Then

$$
\begin{aligned}
& v(i,j+1) - v(i+1,j) - v(i+1,j+1) + v(i+2,j) \\
\geq\ & v(i,j+1) - v^{\phi_2}(i+1,j) - v^{\phi_3}(i+1,j+1) + v(i+2,j) \\
=\ & E\int_0^{\beta} e^{-\alpha t}[h(X_t^4-2, Y_t^4+1) - h(X_t^4-1, Y_t^4) - h(X_t^4-1, Y_t^4+1) + h(X_t^4, Y_t^4)]dt \\
& + Ee^{-\alpha\beta}(-r_2 + r_1 + r_1 - r_1) + E\int_{\beta}^{\tau} e^{-\alpha t}(h_1 - h_2)dt + Ee^{-\alpha\tau}(r_2 - r_1) \\
& + Ee^{-\alpha\tau}\begin{cases} [v(0,0) - v(0,0) - v(0,0) + v(0,0)], \text{if } Y_{\tau-}^1 = Y_{\tau-}^2 = 0 \\ \\ [v(0, Y_\tau^4 - 1) - v(0, Y_\tau^4 - 1) - v(0, Y_\tau^4) + v(0, Y_\tau^4)], \text{o.w.} \end{cases}
\end{aligned}
$$

The first term is 0 because of the linear holding cost rate. Using the fact that $h_1 \geq h_2$ and $\tau \geq \beta$, one can show that (3.9) holds.

Case b.2: $\tau = \tau_2$. Then

$$
\begin{aligned}
& v(i,j+1) - v(i+1,j) - v(i+1,j+1) + v(i+2,j) \\
\geq\ & v(i,j+1) - v^{\phi_2}(i+1,j) - v^{\phi_3}(i+1,j+1) + v(i+2,j) \\
=\ & E\int_0^{\tau} e^{-\alpha t}[h(X_t^4-2, Y_t^4+1) - h(X_t^4-1, Y_t^4) - h(X_t^4-1, Y_t^4+1) + h(X_t^4, Y_t^4)]dt \\
& + Ee^{-\alpha\tau}(v(X_\tau^1, Y_\tau^1) - v(X_\tau^2, Y_\tau^2) - v(X_\tau^3, Y_\tau^3) + v(X_\tau^4, Y_\tau^4)).
\end{aligned}
$$

We have the following possibilities.

Case b.2.1: A class 1 arrival is accepted by Process 1 and rejected by Process 4. Let Process 2 accept the arrival and Process 3 reject. Then the states in four processes at τ are $(X_\tau^4 - 1, Y_\tau^4 + 1)$, (X_τ^4, Y_τ^4), $(X_\tau^4 - 1, Y_\tau^4 + 1)$, and (X_τ^4, Y_τ^4), respectively. Note that Process 1 and 3 couple, so do Process 2 and 4. So (3.9) holds.

Case b.2.2: A class 1 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 accept the arrival and Process 3 reject. Then the states in four processes at τ are $(X_\tau^4 - 3, Y_\tau^4 + 1)$, $(X_\tau^4 - 1, Y_\tau^4)$, $(X_\tau^4 - 2, Y_\tau^4 + 1)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 - 2, Y_\tau^4) + v(X_\tau^4 - 2, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4 + 1)$, we have

$$
\begin{aligned}
B &= v(X_\tau^4 - 3, Y_\tau^4 + 1) - v(X_\tau^4 - 2, Y_\tau^4) - v(X_\tau^4 - 2, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4) \\
&\quad + v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4) - v(X_\tau^4 - 1, Y_\tau^4 + 1) + v(X_\tau^4, Y_\tau^4) \\
&\quad + v(X_\tau^4 - 2, Y_\tau^4) - v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4) + v(X_\tau^4 - 1, Y_\tau^4 + 1).
\end{aligned}
$$

Note that the first four terms and the second four terms are inequality (3.9) evaluated at $(X_\tau^4 - 3, Y_\tau^4)$ and $(X_\tau^4 - 2, Y_\tau^4)$, respectively, so the above argument can be repeated until Case b.1 or Case b.2.1 happens. The last four terms are inequality (3.4) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, which is non-negative by Lemma 10.

Case b.2.3: A class 2 arrival is accepted by Process 1 and rejected by Process 4. Let Process 2 accept the arrival and Process 3 reject. Then the states in four processes at τ are $(X_\tau^4 - 2, Y_\tau^4 + 2)$, $(X_\tau^4 - 1, Y_\tau^4 + 1)$, $(X_\tau^4 - 1, Y_\tau^4 + 1)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 - 2, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4)$, we have

$$
\begin{aligned}
B &= v(X_\tau^4 - 2, Y_\tau^4 + 2) - v(X_\tau^4 - 1, Y_\tau^4 + 1) - v(X_\tau^4 - 2, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4) \\
&\quad + v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4) - v(X_\tau^4 - 1, Y_\tau^4 + 1) + v(X_\tau^4, Y_\tau^4).
\end{aligned}
$$

Note that the first four terms are inequality (3.8) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, so the argument in part (a) can be repeated until Case a.1 or Case a.2.4 happens. The second four terms are inequality (3.9) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, so the above argument can be repeated until Case b.1 or Case b.2.1 happens.

Case b.2.4: A class 2 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 accept the arrival and Process 3 reject. Then the states in four processes at τ are $(X_\tau^4 - 2, Y_\tau^4)$, $(X_\tau^4 - 1, Y_\tau^4)$, $(X_\tau^4 - 1, Y_\tau^4)$, and (X_τ^4, Y_τ^4), respectively. Adding and subtracting $v(X_\tau^4 - 2, Y_\tau^4 + 1) + v(X_\tau^4 - 1, Y_\tau^4 + 1)$, we have

$$\begin{aligned} B &= v(X_\tau^4 - 2, Y_\tau^4) - v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4) + v(X_\tau^4 - 1, Y_\tau^4 + 1) \\ &\quad + v(X_\tau^4 - 2, Y_\tau^4 + 1) - v(X_\tau^4 - 1, Y_\tau^4) - v(X_\tau^4 - 1, Y_\tau^4 + 1) + v(X_\tau^4, Y_\tau^4). \end{aligned}$$

Note that the first four terms are inequality (3.4) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, which is non-negative by Lemma 10. The second four terms are inequality (3.9) evaluated at $(X_\tau^4 - 2, Y_\tau^4)$, so the above argument can be repeated until Case b.1 or Case b.2.1 happens. □

Corollary 1. *If $h_1 \geq h_2$ and $r_1 \geq r_2$, then v is convex in both i and j, i.e.,*

$$v(i+2, j) - v(i+1, j) \geq v(i+1, j) - v(i, j), \tag{3.10}$$

$$v(i, j+2) - v(i, j+1) \geq v(i, j+1) - v(i, j). \tag{3.11}$$

Proof. (3.10) is implied by (3.4) and (3.9), and (3.11) is implied by (3.4) and (3.8). □

Theorem 12. *The socially optimal policy for admitting class 2 customers is characterized by a monotonically decreasing switching curve, i.e., for each $i \geq 0$, there exists a threshold $L_2^s(i)$, such that a class 2 arrival in state (i, j) is accepted if and only if $j < L_2^s(i)$. Furthermore, $L_2^s(i)$ is monotonically decreasing in i.*

Proof. Using supermodularity and convexity in j, one can prove this theorem by following similar argument as in the proof for Theorem 11. □

3.4 Class Optimization

We consider class-optimal policies in this section. The objective of a class-optimal policy for class k, $k = 1, 2$, is to minimize the expected total discounted net cost generated by all customers in class k.

3.4.1 Optimal Policies for Class 1

We consider optimal admission control policies for class 1 customers first. Denote $v(i)$ the expected total discounted net cost generated by a class-optimal policy for class 1 over an infinite horizon starting from state i, where i is the number of class 1 customers in the system. Note that class 1 customers don't see class 2 customers under class optimization because of their higher priority. Thus, after uniformizing, the optimality equation can be written as

$$v(i) = h_1 i + \lambda_1 \min\{v(i), v(i+1)\} + \mu v((i-1)^+ - r_1 I_{\{i \geq 1\}}). \tag{3.12}$$

Lemma 13. $v(1) - v(0) + r_1 \geq 0$.

Proof. Define two processes on the same probability space so that they see the same arrivals and potential services. Process 1 starts with 1 class 1 customer and follows optimal policy. Process 2 starts with 0 class 1 customers and follows policy ϕ which is described below. Let τ be the first time Process 1 has 0 class 1 customers. Let Process 2 take the same action as Process 1 upon each arrival until time τ, then follow the optimal policy afterwards. Therefore, Process 1 has one more class 1 customer than Process 2 until time τ. Two processes become identical from then on. Thus,

$$\begin{aligned} v(1)-v(0) &\geq v(1)-v^{\phi}(0) \\ &= E\int_0^{\tau} e^{-\alpha t}h_1 dt + Ee^{-\alpha\tau}(-r_1+v(0)-v(0)) \\ &\geq -r_1 Ee^{-\alpha\tau} \geq -r_1. \end{aligned}$$

$\square$

Lemma 14. *v is convex, i.e.,*

$$v(i+2)-v(i+1)-v(i+1)+v(i) \geq 0. \tag{3.13}$$

Proof. Define four processes on the same probability space so that they see the same arrivals and potential services. Process 1 and 4 follow optimal policies and start in state $i+2$ and i, respectively. Process 2 and 3 start in state $i+1$ and use policies ϕ_2 and ϕ_3, respectively, which are described below. Denote the state of Process k at time t by (X_t^k, Y_t^k), $k=1,2,3,4$.

Let τ_1 be the first time Process 2 and 3 have 0 class 1 customers. Let τ_2 be the first time Process 1 and 4 take different actions. Define $\tau=\min\{\tau_1,\tau_2\}$. Let Process 2 and 3 take the same action as Process 1 and 4 upon each arrival until time τ, then follow the optimal policy afterwards. Thus

$$\begin{aligned} & v(i+2)-v(i+1)-v(i+1)+v(i) \\ \geq\ & v(i+2)-v^{\phi_2}(i+1)-v^{\phi_3}(i+1)+v(i) \\ =\ & E\int_0^{\tau} e^{-\alpha t}[h(X_t^4+2)-h(X_t^4+1)-h(X_t^4+1)+h(X_t^4)]dt \\ & +Ee^{-\alpha\tau}(-R_1+R_2+R_3-R_4)+Ee^{-\alpha\tau}(v(X_\tau^1)-v(X_\tau^2)-v(X_\tau^3)+v(X_\tau^4)), \end{aligned}$$

where R_i is the potential reward generated in Process i at time τ. It can be easily seen that the first term is 0 because of the linear holding cost rate.

To simplify notation, define

$$\bar{D} = v(i+2) - v(i+1) - v(i+1) + v(i), \tag{3.14}$$

$$\bar{B} = v(X_\tau^1) - v(X_\tau^2) - v(X_\tau^3) + v(X_\tau^4). \tag{3.15}$$

Also define A as in (3.6).

Case 1: $\tau = \tau_1$. Then the states in four processes at τ are 1, 0, 0, 0, respectively. The rewards generated at τ are $R_1 = R_2 = R_3 = r_1$, and $R_4 = 0$. Therefore,

$$\bar{D} \geq Ee^{-\alpha\tau}(v(1) - v(0) + r_1) \geq 0,$$

where the last inequality follows from Lemma 13.

Case 2: $\tau = \tau_2$. Then $A = 0$. We have the following possibilities.

Case 2.1: A class 1 arrival is accepted by Process 1 and rejected by Process 4. Let Process 2 accept and Process 3 reject the arrival. Then the states in four processes at τ are $X_\tau^4+3, X_\tau^4+2, X_\tau^4+1, X_\tau^4$, respectively. Adding and subtracting $v(X_\tau^4+1)+v(X_\tau^4+2)$, we have

$$\begin{aligned}\bar{B} &= v(X_\tau^4+3) - v(X_\tau^4+2) - v(X_\tau^4+2) + v(X_\tau^4+1) \\ &\quad + v(X_\tau^4+2) - v(X_\tau^4+1) - v(X_\tau^4+1) + v(X_\tau^4).\end{aligned}$$

Note that the first four terms and the second four terms are inequality (3.13) evaluated at X_τ^4+1 and X_τ^4, respectively. So the above argument can be repeated until Case 1 or Case 2.2 happens.

Case 2.2: A class 1 arrival is rejected by Process 1 and accepted by Process 4. Let Process 2 accept and Process 3 reject the arrival. Then the states in four processes at τ are $X_\tau^4+2, X_\tau^4+2, X_\tau^4+1, X_\tau^4+1$, respectively. Notice that Process 1 and 2 couple, so do Process 3 and 4. So $\bar{B} = 0$ and hence (3.13) holds. $\square$

Theorem 13. *The class-optimal policy for admitting class 1 customers is characterized by a critical number, i.e., there exists a threshold L_1^c, such that a class 1 arrival in state i is accepted if and only if $i < L_1^c$.*

Proof. Define

$$L_1^c = \min\{i : v(i+1) > v(i)\}.$$

Using Lemma 14 one can easily show that a class 1 arrival is accepted if and only if $i < L_1^c$. $\square$

3.4.2 Optimal Policies for Class 2

We consider optimal admission control policies for class 2 customers next. Denote $v(i,j)$ the expected total discounted net cost generated by a class-optimal policy for class 2 over an infinite horizon starting from state (i,j). Assuming class 1 customers are admitted according to the class-optimal policy for class 1, the optimality equation can be written as

$$\begin{aligned} v(i,j) &= h_2 j + \lambda_1 \begin{cases} v(i+1,j), & \text{if } \ i < L_1^c \\ v(i,j), & \text{if } \ i >= L_1^c \end{cases} + \lambda_2 \min\{v(i,j+1), v(i,j)\} \\ &\quad + \mu \begin{cases} v(i-1,j), & \text{if } \ i \geq 1 \\ v(0,j-1) - r_2, & \text{if } \ i = 0, j \geq 1 \\ v(0,0), & \text{if } \ i = 0, j = 0. \end{cases} \end{aligned} \tag{3.16}$$

Lemma 15. $v(0,1) - v(0,0) + r_2 \geq 0$.

Proof. Same argument as in the proof for Lemma 9 applies. $\square$

Lemma 16. *v is convex in j, i.e.,*

$$v(i,j+2) - v(i,j+1) - v(i,j+1) + v(i,j) \geq 0. \tag{3.17}$$

Proof. Same argument as in the proof for Lemma 14 applies after the following changes. Replace class 1 by class 2. Replace $v(i)$ by $v(i, j)$, $v(i+1)$ by $v(i, j+1)$, etc. Replace r_1 by r_2. □

Lemma 17. *v is supermodular, i.e.,*

$$v(i+1, j+1) - v(i, j+1) - v(i+1, j) + v(i, j) \geq 0. \tag{3.18}$$

Proof. Same argument as in the proof for Lemma 10 applies after the following changes. No reward is generated when a class 1 customer finishes service, i.e., $r_1 = 0$. Case 2.1 does not exist, since a class 1 arrival is always accepted in state (i, j) if it is accepted in state $(i+1, j)$. Case 2.2 is the same as in Lemma 10 except that it only happens when $i = L_1^c - 1$. □

Theorem 14. *The class-optimal policy for admitting class 2 customers is characterized by a monotonically decreasing switching curve, i.e., for each $i \geq 0$, there exists a threshold $L_2^c(i)$, such that a class 2 arrival in state (i, j) is accepted if and only if $j < L_2^c(i)$. Furthermore, $L_2^c(i)$ is monotonically decreasing in i.*

Proof. Using supermodularity and convexity in j, one can prove this theorem by following similar argument as in the proof for Theorem 11. □

3.5 Numerical Comparison

We compare policies under different criteria numerically in this section. The numerical examples are computed by truncating the state space and using standard value iteration algorithm as described in Section 2.6.

Figure 3.1 plots the cost of class-optimal policy for class 1 customers against i, the number of class 1 customers in starting state. Figure 3.2 plots the cost of class-optimal policy for class 2 customers against j, the number of class 2 customers in starting state, for different i. Figure 3.1 and 3.2 use the following parameters $\alpha = 0.2, \mu = 0.5, \lambda_1 = 0.15, \lambda_2 = 0.15, h_1 = 0.3, h_2 = 0.2, r_1 = 25, r_2 = 18$. Note that the class-optimal value function is not monotone in i or j in this case, while the monotonicities ((2.12), (2.13)) hold for the case where the reward is generated at the time of joining the queue.

Figure 3.3 and 3.4 plot the cost of socially optimal policy against i and j for fixed j and i, respectively. They use the same parameters as in Figure 3.1 and 3.2 except that $r_1 = 15, r_2 = 10$. The socially optimal value function is not monotone in i or j as contrary to the model discussed in Chapter 2. So moving the reward time changes the nature of the problem.

Figure 3.5 and 3.6 plot the switching curves under three optimization criteria for class 1 and class 2, respectively. Figure 3.5 uses the following parameters $\alpha = 0.1, \mu = 0.5, \lambda_1 = 0.1, \lambda_2 = 0.3, h_1 = 25, h_2 = 20, r_1 = 450, r_2 = 300$. Figure 3.6 uses the following parameters $\alpha = 0.1, \mu = 0.5, \lambda_1 = 0.39, \lambda_2 = 0.01, h_1 = 2, h_2 = 0.3, r_1 = 550, r_2 = 500$.

Note that for class 1 (higher priority) customer, individually optimal policy accepts the most and socially optimal policy accepts the least number of customers. For class 2 (lower priority) customer, socially optimal policy accepts more customers than class optimal policy, which is the exact opposite to the comparison result for class 1. Depending on the parameters, individually optimal policy can accept either more or fewer customers than either of the other two policies. The above observations agree with the results we obtained for the previous model. The intuitive explanation provided for the previous model also applies here.

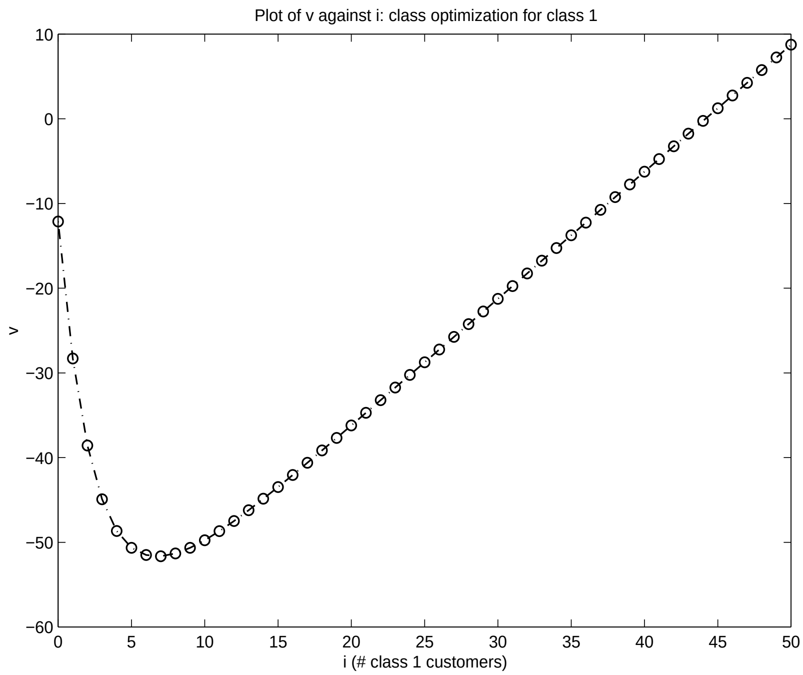

Figure 3.1: Class optimization for class 1: v against i

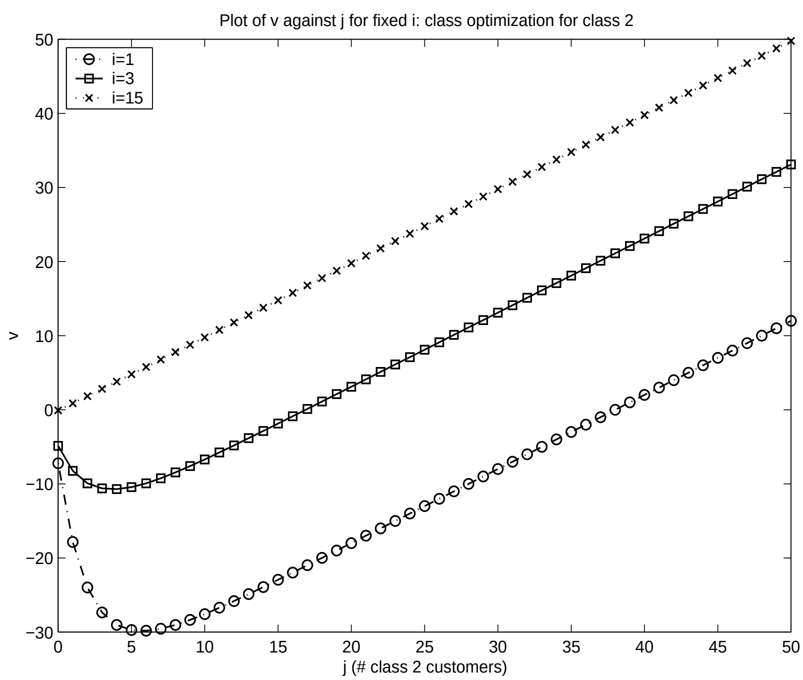

Figure 3.2: Class optimization for class 2: v against j for fixed i

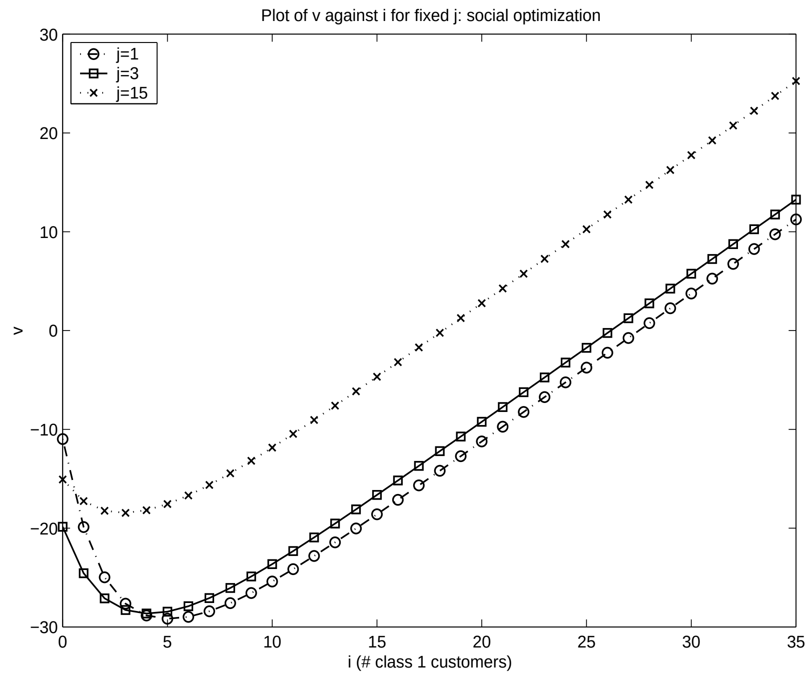

Figure 3.3: Social optimization: v against i for fixed j

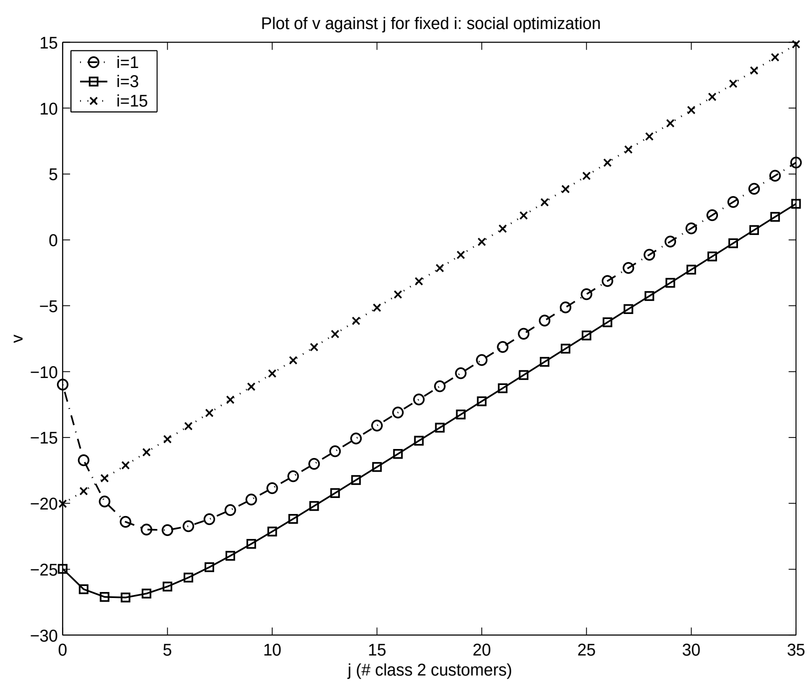

Figure 3.4: Social optimization: v against j for fixed i

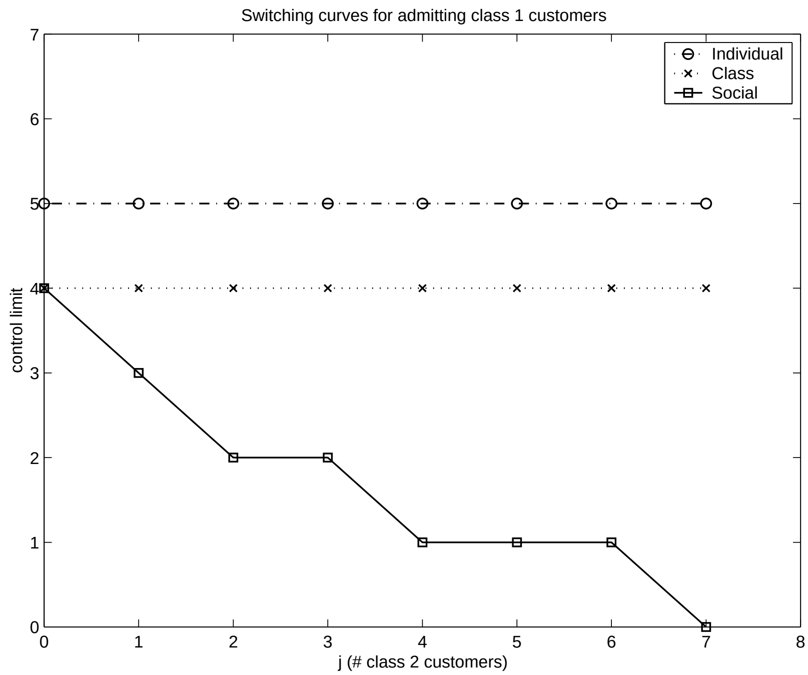

Figure 3.5: Class 1 switching curves

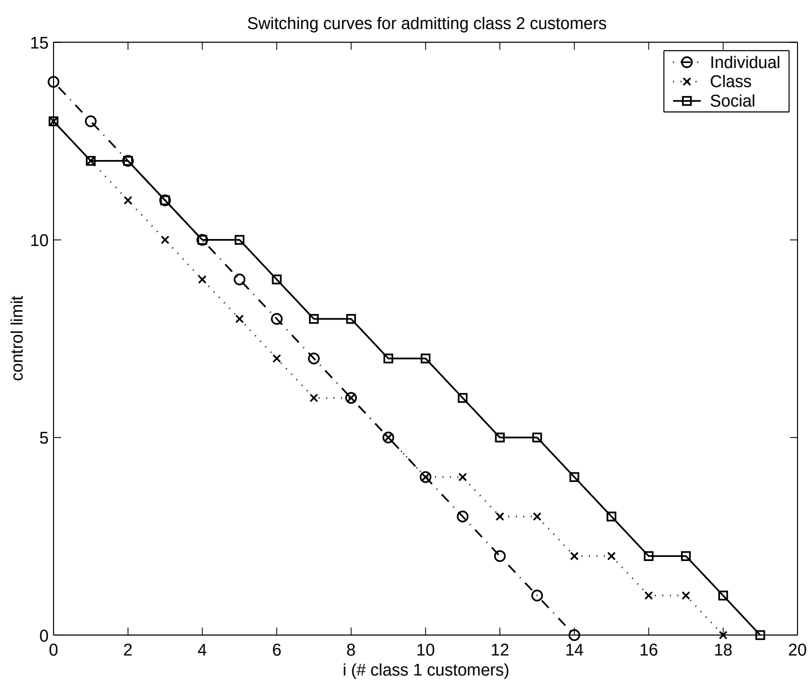

Figure 3.6: Class 2 switching curves

Chapter 4

Dynamic Routing

4.1 Problem Description

In this chapter, we consider the dynamic warranty repair allocation problem. Assume sales of class k items form a Poisson process with rate λ_k, denoted by $PP(\lambda_k)$, $k = 1, 2$. Warranty length for either class is a constant W. The manufacturer outsources the warranty repairs to V vendors (one of them could be the manufacturer's facility itself). The life times of the items are i.i.d. $\exp(\beta)$ random variables. When an item fails while it is under warranty, it is sent to one of the V vendors for repair. There is one repair person at each vendor. The repair times are i.i.d. $\exp(\mu_i)$ at vendor i (same for both classes). Class 1 items have preemptive resume priority over class 2 items in repair service. The manufacturer pays vendor i a fixed fee c_i each time a repair is assigned to vendor i, $i = 1, \cdots, V$. While a class k item is awaiting or under repair at vendor i, the manufacturer incurs a holding cost (good will cost) at rate h_{ki}, $k = 1, 2, i = 1, \cdots, V$. We assume items covered by higher priority warranty generate holding cost at a higher rate, i.e., $h_{1i} \geq h_{2i}$, $i = 1, \cdots, V$. This situation agrees with the way the well-known c-μ rule assigns priorities to multiple classes of jobs at a single service station, i.e, higher priority is given to the class with larger c-μ

ratio (c is the holding cost rate in our case). Items are as good as new after repair. The goal of the manufacturer is to assign repairs to vendors in such a way that the expected long-run average cost is minimized.

The complexity of the problem prevents us from finding optimal policies. Hence we turn our attention to heuristic allocation procedures. One natural way of obtaining an approximate solution is to simplify the problem by assuming exponential warranty length and formulate it as an Markov decision process (MDP). The optimal policy for the resulting MDP can be expected to work reasonably well for the original problem. However, the curse of dimensionality (of the state space) makes solving the Bellman equations of the MDP impractical even for small-size problems. We present four heuristics that are applicable to large problems. Among the four heuristics, the Generalized Join the Shortest Queue (GJSQ) policy is of our primary interest. The GJSQ policy is derived by applying a single policy improvement step to an judiciously chosen initial state-independent policy. We first develop the GJSQ policy then evaluate and compare it with other heuristics using simulation.

4.2 Heuristics

4.2.1 Optimal State-Independent Policy (OSI)

We first consider state-independent policies, i.e., stationary policies that do not depend on the real-time system state. We confine ourselves to a specific, yet natural, type of state-independent policy, namely, a Bernoulli splitting policy. Under this policy, a type k repair is assigned to vendor i with probability p_{ki}, where $\sum_{i=1}^{V} p_{ki} = 1$, $k = 1, 2$. Let $\boldsymbol{p}_k = (p_{k1}, p_{k2}, \cdots, p_{kV})$, $k = 1, 2$. Then the Bernoulli splitting policy can be denoted by $(\boldsymbol{p}_1, \boldsymbol{p}_2)$. We aim to find an optimal Bernoulli splitting policy that minimizes the long-run average cost. In order to compute the long-run average cost of a Bernoulli splitting policy with splitting probabilities $(\boldsymbol{p}_1, \boldsymbol{p}_2)$, we simplify the real system by assuming that failures of type k items under warranty occur according to a $PP(\phi_k)$, where $\phi_k = \lambda_k W\beta$ (or, equivalently, the number of type k functioning items under warranty is a constant $\lambda_k W$). Then type k repairs arrive at vendor i according to a $PP(\phi_k p_{ki})$. We assume $\sum_{i=1}^{V} \mu_i > \phi_1 + \phi_2$, i.e., the total arrival rate of failed items is less than the total service rate. As a result, there must exist policies $(\boldsymbol{p}_1, \boldsymbol{p}_2)$ such that $\phi_1 p_{1i} + \phi_2 p_{2i} < \mu_i$, $i = 1, ..., V$. We only consider such stable policies for the rest of the paper.

Because of their preemptive resume priority, type 1 items simply do not see type 2 items in the repair queue. So the expected number of type 1 items at vendor i is

$$L_{1i}(p_{1i}) = \phi_1 p_{1i}/(\mu_i - \phi_1 p_{1i}). \tag{4.1}$$

Obviously, the expected number of all items at vendor i is $(\phi_1 p_{1i}+\phi_2 p_{2i})/(\mu_i-(\phi_1 p_{1i}+\phi_2 p_{2i}))$. Hence the expected number of type 2 items at vendor i is

$$L_{2i}(p_{1i}, p_{2i}) = \frac{\phi_1 p_{1i} + \phi_2 p_{2i}}{\mu_i - (\phi_1 p_{1i} + \phi_2 p_{2i})} - \frac{\phi_1 p_{1i}}{\mu_i - \phi_1 p_{1i}}. \tag{4.2}$$

Let

$$f_{1i}(x) = \begin{cases} (h_{1i} - h_{2i})\frac{x}{\mu_i - x}, & \text{if } x < \mu_i \\ \infty, & \text{if } x \geq \mu_i, \end{cases} \tag{4.3}$$

$$f_{2i}(x) = \begin{cases} c_i x + h_{2i}\frac{x}{\mu_i - x}, & \text{if } x < \mu_i \\ \infty, & \text{if } x \geq \mu_i, \end{cases} \tag{4.4}$$

and

$$f_i(x_1, x_2) = f_{1i}(x_1) + f_{2i}(x_1 + x_2). \tag{4.5}$$

Then the long-run average cost rate at vendor i is $f_i(\phi_1 p_{1i}, \phi_2 p_{2i})$.

Therefore, the optimal Bernoulli splitting policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ can be obtained by solving the following optimization problem:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{V} f_i(\phi_1 p_{1i}, \phi_2 p_{2i}) \\
\text{s.t.} \quad & \sum_{i=1}^{V} p_{ki} = 1, \quad k = 1, 2 \\
& p_{ki} \geq 0, \quad k = 1, 2, \quad i = 1, 2, \cdots, V.
\end{aligned} \tag{4.6}
$$

Note that the objective function is separable in terms of pairs (p_{1i}, p_{2i}), i.e., it is a sum of functions of two variables (p_{1i}, p_{2i}) each. Each single term can be further decomposed as in (4.5). To take advantage of the structure of this problem and apply simple and efficient algorithm, we solve the discretized version of the above optimization problem as described in the following. Suppose $\lambda_k W$ are integers, $k = 1, 2$. Otherwise, take their integer parts. Associate a pair of integers (y_{1i}, y_{2i}) with each vendor, where $\sum_{i=1}^{V} y_{ki} = \lambda_k W$, and let $p_{ki} = \frac{y_{ki}}{\lambda_k W}$, $k = 1, 2, i = 1, 2, \cdots, V$. This can be interpreted in the following way: Assume that there are $\lambda_k W$ type k items under warranty. Assign y_{ki} of them to vendor i and always send them to vendor i for repair upon failure. In terms of (y_{1i}, y_{2i}), the long-run average cost rate at vendor i is $f_i(y_{1i}\beta, y_{2i}\beta)$, where f_i is defined in (4.5).

Therefore, the discretized version of (4.6) can be written as

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{V} f_{1i}(y_{1i}\beta) + f_{2i}(y_{1i}\beta + y_{2i}\beta) \\
\text{s.t.} \quad & \sum_{i=1}^{V} y_{ki} = \lambda_k W, \quad k = 1, 2 \\
& y_{ki} \geq 0 \ \text{ and integer}, \quad k = 1, 2, \quad i = 1, 2, \cdots, V.
\end{aligned} \tag{4.7}
$$

Optimization problem (4.7) can be formulated as a minimum cost network flow problem which can be solved by a *Successive Shortest Path Algorithm* with complexity $O(V + (\lambda_1 + \lambda_2) W \log V)$ (see Buczkowski et al. [9], and Ahuja et al. [2]). We provide

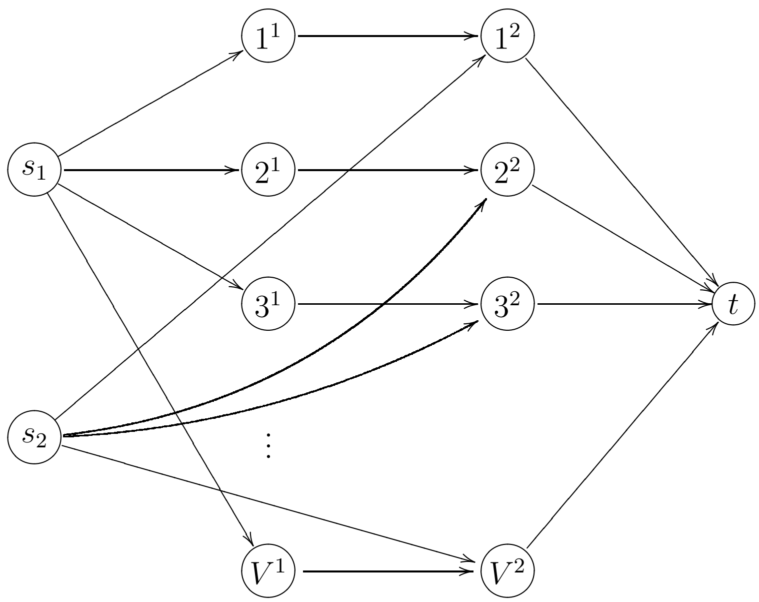

Figure 4.1: Network model of two-priority problem

the formulation and algorithm here for ready reference.

Figure 4.1 shows the network model for (4.7). There are two source nodes s_1, s_2 with supplies of $\lambda_1 W$ and $\lambda_2 W$, respectively. There is one sink node t with a demand of $(\lambda_1+\lambda_2)W$. All other nodes are transshipment nodes with 0 demand and 0 supply. The arc properties are summarized in table 4.1.

Arc	Capacity	Flow	Cost
(s_i, j^i)	$\lambda_i W$	y_{ij}	0
(j^1, j^2)	$\lambda_1 W$	y_{1j}	$f_{1j}(y_{1j}\beta)$
(j^2, t)	$(\lambda_1+\lambda_2)W$	$y_{1j}+y_{2j}$	$f_{2j}((y_{1j}+y_{2j})\beta)$

Table 4.1: Arc properties for the network representation of (4.7)

Define $\boldsymbol{y}_k = (y_{k1}, \cdots, y_{kV}), k = 1, 2$, $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2)$ and $\delta f(y\beta) = f(y\beta) - f((y-1)\beta), y \geq 1$. The following algorithm can be used to solve the network problem.

Successive Shortest Path Algorithm:

Initialize $\boldsymbol{y} := 0$;

while $\sum_{i=1}^{V} y_{2i} < \lambda_2 W$ do

- compute $\min_{j=1,\ldots,V} \delta f_{2j}((y_{2j}+1)\beta)$,
- increment y_{2k} by 1, where $k \in \arg\min_{j=1,\ldots,V} \delta f_{2j}((y_{2j}+1)\beta)$;

end

while $\sum_{i=1}^{V} y_{1i} < \lambda_1 W$ do

- compute $d_i = \delta f_{1i}((y_{1i}+1)\beta) + \delta f_{2i}((y_{1i}+y_{2i}+1)\beta), i = 1, ..., V$ and $d_{V+1} = \min_{y_{2j}>0} \delta f_{1j}((y_{1j}+1)\beta) + \min_{j=1,\ldots,V} \delta f_{2j}((y_{1j}+y_{2j}+1)\beta)$.
- let $q \in \arg\min_{j=1,..,V+1} d_j$.
  If $q \in \{1, ..., V\}$, increment y_{1q} by 1.
  If $q = V+1$, let $k \in \arg\min_{y_{2j}>0} \delta f_{1j}((y_{1j}+1)\beta)$ and $p \in \arg\min_{j=1,\ldots,V} \delta f_{2j}((y_{1j}+y_{2j}+1)\beta)$. Increment y_{1k} and y_{2p} by 1 and decrement y_{2k} by 1 unit.

end

Denote the optimal solution to (4.7) by (y_{1i}^*, y_{2i}^*), then we have the following approximate solution to (4.6):

$$p_{1i}^* = \frac{y_{1i}^*}{\lambda_1 W}, \quad p_{2i}^* = \frac{y_{2i}^*}{\lambda_2 W}, \quad i = 1, 2, \cdots, V. \tag{4.8}$$

Note that p_{ki}^* is an optimal solution to (4.6) to a degree of accuracy of $\frac{1}{\lambda_k W}$, $k = 1, 2$, $i = 1, \ldots, V$. The expected numbers of items under warranty, $\lambda_k W$, are usually large in real problems, in which case, the discretized solution is very close to the real-valued optimal solution.

4.2.2 Generalized Join the Shortest Queue Policy (GJSQ)

We continue to assume that the number of type k functioning items under warranty is a constant $\lambda_k W$ and failures of type k items under warranty occur according to a $PP(\phi_k)$, where $\phi_k = \lambda_k W\beta$, $k = 1, 2$. Therefore, the original warranty repair allocation problem reduces to the problem of routing items arriving according to two independent Poisson streams to several vendors where service is provided according to a predetermined priority policy.

A generalized model of this situation is studied by Ansell et al. [3]. In their model, jobs from a number of different classes arrive according to independent Poisson processes. Jobs are either generic or dedicated, and they are routed to a set of service stations. Dedicated jobs can be processed only by a specified station, while generic jobs can be processed at any station. Jobs are served according to a static priority policy at each station. A holding cost is incurred at a class-dependent rate while a job is in the system. The objective is to minimize the long-run average holding cost rate. The authors develop a dynamic routing heuristic by applying a single policy improvement step to an initial static policy (see also Krishnan [23] and Tijms [45] for this approach). They name the resulting index-based heuristic "Generalized Join the Shortest Queue" (GJSQ) policy. We provide their main result here for ready reference.

Denote the set of generic jobs by G and the set of dedicated jobs to station i by D_i. Denote the number of class k jobs that are currently awaiting or under service at station i by x_{ki}, $k \in G \cup D_i$, $i = 1, \cdots, V$. Let $\boldsymbol{x}^i = \{x_{ki} | k \in G \cup D_i\}$. For each class k, the GJSQ policy associates with each station i an index I_{ki} which is a linear function of the number of jobs of each class at station i, i.e.,

$$I_{ki}(\boldsymbol{x}^i) = \sum_{l \in G \cup D_i} \theta^i_{kl} x_{li} + \delta^i_k, \quad k \in G \cup D_i, \quad i = 1, \cdots, V,$$

where the coefficients θ^i_{kl} and δ^i_k are constants. The GJSQ policy routes an incoming class k job to the station with the smallest index.

Although structurally the simplified version of our problem is a special case of that studied by Ansell et al. [3] (we consider two generic classes and no dedicated classes), our approach differs from theirs in the following ways: In addition to holding cost, we also allow a station-dependent fixed cost per assignment, which is not considered by Ansell et al. [3]. Furthermore, we are able to give tractable closed-form expressions for the coefficients following complicated queueing theoretic calculations, while in Ansell et al. [3] the coefficients are given as a solution to an infinite set of recursive equations.

We use the solution $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ given in (4.8) as the initial state-independent policy. Following Ansell et al. [3], we show that the linear structure of the indices continues to hold in the presence of fixed costs. In particular, for our problem there exist two indices $I_{1j}(x_{1j}, x_{2j})$ and $I_{2j}(x_{1j}, x_{2j})$ for each vendor $j = 1, \ldots, V$ of the following form

$$I_{1j}(x_{1j}, x_{2j}) = A_{1j} + B_{1j}x_{1j} + C_{1j}x_{2j}, \tag{4.9}$$

$$I_{2j}(x_{1j}, x_{2j}) = A_{2j} + B_{2j}x_{1j} + C_{2j}x_{2j}. \tag{4.10}$$

After some lengthy algebra, we get the closed-form expressions for the coefficients of the indices. We introduce the following notations before stating the theorem:

$$\phi_{kj} = \phi_k p_{kj}^*, \tag{4.11}$$

$$\eta_j = \frac{1}{\mu_j - \phi_{1j}}, \tag{4.12}$$

$$\xi_j = \frac{1}{\mu_j - \phi_{1j} - \phi_{2j}}, \tag{4.13}$$

where $k = 1, 2$, $j = 1, \ldots, V$.

Theorem 15. *Assume $\mu_j > \phi_{1j} + \phi_{2j}$ and let f_j, ϕ_{kj}, η_j and ξ_j be as given in (4.5), (4.11), (4.12), and (4.13), respectively. Then the coefficients of the indices defined in*

(4.9) and (4.10) are given by

$$\begin{aligned}
A_{1j} &= c_j + [c_j(\phi_{1j}+\phi_{2j}) + h_{1j}]\eta_j + f_j(\phi_{1j},\phi_{2j})\xi_j + (\phi_{1j}h_{1j} + \frac{1}{2}\phi_{2j}h_{2j})\eta_j^2 \\
&\quad + [c_j(\phi_{1j}+\phi_{2j})\phi_{2j} + \phi_{2j}h_{2j}]\eta_j\xi_j + \frac{1}{2}(\phi_{1j}+\mu_j)\phi_{2j}h_{2j}\eta_j^3 \\
&\quad + (\phi_{1j}\phi_{2j}h_{1j} + \frac{1}{2}\phi_{2j}^2 h_{2j})\eta_j^2\xi_j + \frac{1}{2}(\phi_{1j}+\mu_j)\phi_{2j}^2 h_2\eta_j^3\xi_j + \phi_{2j}^2\mu_j h_{2j}\eta_j^2\xi_j^2, \\
B_{1j} &= h_{1j}\eta_j + \phi_{2j}h_{2j}\eta_j^2 + \phi_{2j}^2 h_{2j}\eta_j^2\xi_j, \\
C_{1j} &= h_{2j}\eta_j + \phi_{2j}h_{2j}\eta_j\xi_j, \\
A_{2j} &= c_j + [c_j(\phi_{1j}+\phi_{2j}) + h_{2j} + f_j(\phi_{1j},\phi_{2j})]\xi_j + \phi_{1j}h_{1j}\eta_j\xi_j + \phi_{2j}\mu_j h_{2j}\eta_j\xi_j^2, \\
B_{2j} &= h_{2j}\eta_j + \phi_{2j}h_{2j}\eta_j\xi_j, \\
C_{2j} &= h_{2j}\xi_j.
\end{aligned}$$

Proof. Assume that the number of type k functioning items under warranty is a constant $\lambda_k W$ and failures of type k items under warranty occur according to a $PP(\phi_k)$, where $\phi_k = \lambda_k W\beta, k = 1, 2$. Denote by g^* the long-run average cost rate incurred by the manufacturer under policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$. Denote by $\omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2)$ the bias associated with policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ starting from state $(\boldsymbol{x}_1, \boldsymbol{x}_2)$. Rescaling the time scale so that $\phi_1 + \phi_2 + \sum_{i=1}^V \mu_i = 1$, the optimality equation can be written as

$$\begin{aligned}
& g^* + \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
= \; & \sum_{i=1}^V (h_{1i}x_{1i} + h_{2i}x_{2i}) + \sum_{i=1}^V \begin{cases} \mu_i\omega^*(\boldsymbol{x}_1 - \boldsymbol{e}_i, \boldsymbol{x}_2), & \text{if } x_{1i} \geq 1 \\ \mu_i\omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2 - \boldsymbol{e}_i), & \text{if } x_{1i} = 0, x_{2i} \geq 1 \\ \mu_i\omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2), & \text{if } x_{1i} = 0, x_{2i} = 0 \end{cases} \\
+ \; & \phi_1 \sum_{i=1}^V p_{1i}^*(c_i + \omega^*(\boldsymbol{x}_1 + \boldsymbol{e}_i, \boldsymbol{x}_2)) + \phi_2 \sum_{i=1}^V p_{2i}^*(c_i + \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2 + \boldsymbol{e}_i))
\end{aligned} \tag{4.14}$$

We improve the policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ by applying one step of policy-improvement, which works as follows. When a type k item fails, we send it to the vendor where the cost increment caused by assigning one more type k repair is the smallest assuming that policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ is applied forever afterwards, $k = 1, 2$. Therefore, if a type 1 item fails in state $(\boldsymbol{x}_1, \boldsymbol{x}_2)$, we find vendor index $j^* \in \arg\min_j\{c_j + \omega^*(\boldsymbol{x}_1 + \boldsymbol{e}_j, \boldsymbol{x}_2)\}$ or equivalently $j^* \in \arg\min_j\{c_j + \omega^*(\boldsymbol{x}_1 + \boldsymbol{e}_j, \boldsymbol{x}_2) - \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2)\}$. If a type 2 item fails in state $(\boldsymbol{x}_1, \boldsymbol{x}_2)$, we find vendor index $j^* \in \arg\min_j\{c_j + \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2 + \boldsymbol{e}_j)\}$ or equivalently $j^* \in \arg\min_j\{c_j + \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2 + \boldsymbol{e}_j) - \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2)\}$. Then send the repair to vendor j^*. Next we turn to the computation of $\omega^*(\boldsymbol{x}_1 + \boldsymbol{e}_j, \boldsymbol{x}_2) - \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and $\omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2 + \boldsymbol{e}_j) - \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

Let $v_T(\boldsymbol{x}_1, \boldsymbol{x}_2)$ be the total expected cost of policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ up to time T starting in state $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ at time 0. From Puterman [39], we have

$$\omega^*(\boldsymbol{x}_1 + \boldsymbol{e}_j, \boldsymbol{x}_2) - \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2) = \lim_{T\to\infty}[v_T(\boldsymbol{x}_1 + \boldsymbol{e}_j, \boldsymbol{x}_2) - v_T(\boldsymbol{x}_1, \boldsymbol{x}_2)]. \tag{4.15}$$

Let $v_{iT}(x_{1i}, x_{2i})$ be the total expected cost incurred at vendor i by policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ up to time T. Then we have $v_T(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_{i=1}^{V} v_{iT}(x_{1i}, x_{2i})$.

Denote by g_i^* the expected cost rate at vendor i in steady state under policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$, then

$$g_i^* = f_i(\phi_1 p_{1i}^*, \phi_2 p_{2i}^*), \tag{4.16}$$

where f_i is defined in (4.5). Since the system reaches steady state eventually under any stable policy, $v_{iT}(x_{1i}, x_{2i})$ asymptotically converges to a straight line with slope g_i^* as $T \to \infty$, i.e.,

$$v_{iT}(x_{1i}, x_{2i}) = b_{i,(x_{1i},x_{2i})} + g_i^* T + O(T), \tag{4.17}$$

where $b_{i,(x_{1i},x_{2i})}$ is the intercept of the asymptote and $\lim_{T\to\infty} O(T) = 0$.

Let $T_i(x_{1i}, x_{2i})$ be the time it takes vendor i to reach state $(0, 0)$ for the first time from initial state (x_{1i}, x_{2i}) and let $\tau_i(x_{1i}, x_{2i}) = E[T_i(x_{1i}, x_{2i})]$. Let $J_i(x_{1i}, x_{2i})$ be the expected cost incurred by vendor i starting from initial state (x_{1i}, x_{2i}) until the first time vendor i reaches state $(0, 0)$. Introduce notation $a \wedge b = \min\{a, b\}$. As $T \to \infty$, we have

$$
\begin{aligned}
& v_{iT}(x_{1i}, x_{2i}) \\
= \;& E_{T_i(x_{1i},x_{2i})}[v_{i(T\wedge T_i(x_{1i},x_{2i}))}(x_{1i}, x_{2i})] + E_{T_i(x_{1i},x_{2i})}[v_{i(T-T_i(x_{1i},x_{2i}))}(0,0)] \\
= \;& J_i(x_{1i}, x_{2i}) + O(T) + E_{T_i(x_{1i},x_{2i})}[b_{i,(0,0)} + g_i^*(T - T_i(x_{1i}, x_{2i})) + O(T - T_i(x_{1i}, x_{2i}))] \\
= \;& J_i(x_{1i}, x_{2i}) + v_{iT}(0,0) - g_i^* \tau_i(x_{1i}, x_{2i}) + O(T).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& v_T(\boldsymbol{x}_1 + \boldsymbol{e}_j, \boldsymbol{x}_2) - v_T(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
= \;& \sum_{\substack{i=1 \\ i\neq j}}^{V} v_{iT}(x_{1i}, x_{2i}) + v_{jT}(x_{1j}+1, x_{2j}) - \sum_{i=1}^{V} v_{iT}(x_{1i}, x_{2i}) \\
= \;& v_{jT}(x_{1j}+1, x_{2j}) - v_{jT}(x_{1j}, x_{2j}) \\
= \;& J_j(x_{1j}+1, x_{2j}) + v_{jT}(0,0) - g_j^* \tau_j(x_{1j}+1, x_{2j}) + O(T) \\
& -(J_j(x_{1j}, x_{2j}) + v_{jT}(0,0) - g_j^* \tau_j(x_{1j}, x_{2j}) + O(T)) \\
= \;& J_j(x_{1j}+1, x_{2j}) - J_j(x_{1j}, x_{2j}) - g_j^*[\tau_j(x_{1j}+1, x_{2j}) - \tau_j(x_{1j}, x_{2j})] + O(T).
\end{aligned}
$$

Substituting the above expression in (4.15), we have

$$
\omega^*(\boldsymbol{x}_1+\boldsymbol{e}_j, \boldsymbol{x}_2) - \omega^*(\boldsymbol{x}_1, \boldsymbol{x}_2) = J_j(x_{1j}+1, x_{2j}) - J_j(x_{1j}, x_{2j}) - g_j^*[\tau_j(x_{1j}+1, x_{2j}) - \tau_j(x_{1j}, x_{2j})].
$$

Define the type 1 index at vendor j as

$$
I_{1j}(x_{1j}, x_{2j}) = c_j + J_j(x_{1j}+1, x_{2j}) - J_j(x_{1j}, x_{2j}) - g_j^*[\tau_j(x_{1j}+1, x_{2j}) - \tau_j(x_{1j}, x_{2j})]. \tag{4.18}
$$

Similarly, the type 2 index at vendor j can be defined as

$$I_{2j}(x_{1j}, x_{2j}) = c_j + J_j(x_{1j}, x_{2j}+1) - J_j(x_{1j}, x_{2j}) - g_j^*[\tau_j(x_{1j}, x_{2j}+1) - \tau_j(x_{1j}, x_{2j})]. \tag{4.19}$$

$I_{kj}(x_{1j}, x_{2j})$ can be viewed as the cost increment at vendor j caused by assigning one type k repair to vendor j in state (x_{1j}, x_{2j}) assuming that policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ is applied forever afterwards. When a type k item fails, it is sent to the vendor whose type k index is the smallest.

Next, we derive closed-form expressions for the indices defined in (4.18) and (4.19). Consider a fixed vendor, i.e., vendor i for a fixed i. For notational simplicity, we drop the vendor suffix i. Thus, the single vendor can be viewed as the following queueing system. Two types of failed items arrive at a single-server queue according to $PP(\phi_k)$, $k = 1, 2$, respectively. Repair times are i.i.d. $\exp(\mu)$ for both types of items. Type 1 items have preemptive resume priority in service over type 2 items. Each failed type k item incurs a holding cost at rate h_k throughout its sojourn time at the vendor and a fixed cost c, $k = 1, 2$. We are interested in computing $J(x_1, x_2)$, the expected total cost incurred by this queueing system starting from initial state (x_1, x_2) until the first time it reaches state $(0, 0)$. We use the following notations in the rest of this appendix:

$X_k(t)$ = number of type k items in the system at time t, $k = 1, 2$;

$B_1 = \min\{t \geq 0 : X_1(t) = 0 | X_1(0) = 1\}$, i.e., the busy period for serving type 1 items initiated by a single type 1 item;

$B_2 = \min\{t \geq 0 : X_2(t) = 0 | X_1(0) = 0, X_2(0) = 1\}$, i.e., the busy period for serving both types of items initiated by a single type 2 item;

S_2 = the service completion time of a type 2 item accounting for interruptions from type 1 items. Thus if a type 2 item starts service at time 0, it will complete service at time S_2;

$L_1 = \lim_{t\to\infty} E(X_1(t))$, i.e., the expected number of type 1 items in the system;

$L_{1B} = E[\int_0^{B_1} X_1(t)dt]/E(B_1)$, i.e., the expected number of type 1 items in the system during B_1;

$L_2 = \lim_{t\to\infty} E(X_2(t))$, i.e., the expected number of type 2 items in the system;

C_{11} = the expected holding cost incurred by the type 1 items during B_1;

C_{12} = the expected holding cost incurred by the type 1 items during S_2;

$T_1(x_1) = \min\{t \geq 0 : X_1(t) = 0 | X_1(0) = x_1\}$;

$T(x_1, x_2) = \min\{t \geq 0 : X_1(t) = 0, X_2(t) = 0 | X_1(0) = x_1, X_2(0) = x_2\}$;

$\tau_1(x_1) = E(T_1(x_1))$;

$\tau(x_1, x_2) = E(T(x_1, x_2))$;

$H_1(x_1, x_2)$ = the expected total holding cost incurred by the system starting from initial state (x_1, x_2) until time $T_1(x_1)$;

$H_2(x_1, x_2)$ = the expected total holding cost incurred by the system from time $T_1(x_1)$ until time $T(x_1, x_2)$;

$H(x_1, x_2)$ = the expected total holding cost incurred by the system starting from initial state (x_1, x_2) until time $T(x_1, x_2)$.

Also let $\rho_1 = \frac{\phi_1}{\mu}$, $\quad \rho_2 = \phi_2 E(S_2)$, $\quad \eta = \frac{1}{\mu - \phi_1}$, $\ \xi = \frac{1}{\mu - \phi_1 - \phi_2}$.

Assuming $\phi_1 > 0$ (otherwise, the problem reduces to a single-priority problem), from Prabhu [38] (Chapter 3, Theorem 1) we have

$$E(B_1) = \frac{1}{\mu - \phi_1}, \tag{4.20}$$

and

$$Var(B_1) = \frac{\phi_1 + \mu}{(\mu - \phi_1)^3}. \tag{4.21}$$

Obviously,

$$H(x_1, x_2) = H_1(x_1, x_2) + H_2(x_1, x_2)$$

The following two lemmas compute $H_1(x_1, x_2)$ and $H_2(x_1, x_2)$, respectively.

Lemma 18.

$$H_1(x_1, x_2) = [\frac{1}{2}h_1\eta + \phi_1 h_1 \eta^2 + \frac{1}{2}(\phi_1 + \mu)\phi_2 h_2 \eta^3]x_1 + \frac{1}{2}(h_1\eta + \phi_2 h_2 \eta^2)x_1^2 + h_2\eta x_1 x_2. \tag{4.22}$$

Proof. Since type 1 items do not see type 2 items in the repair queue, we have

$$L_1 = \frac{\rho_1}{1-\rho_1} = \frac{\phi_1}{\mu - \phi_1}. \tag{4.23}$$

Note that ρ_1 is the fraction of time that the server is busy serving type 1 items, therefore

$$L_{1B} = \frac{L_1}{\rho_1} = \frac{\mu}{\mu - \phi_1}. \tag{4.24}$$

By definitions of L_{1B} and B_1, we have

$$C_{11} = h_1 L_{1B} E(B_1), \tag{4.25}$$

where $E(B_1)$ is given by (4.20).

To simplify analysis, we assume that the vendor follows Last-Come-First-Served (LCFS) preemptive service discipline within class 1 items. The assumption of LCFS service discipline is valid because we are interested in total cost, which is independent of the order of service within each class.The assumption of preemption is valid because of the exponential service times. Then $H_1(x_1, x_2)$ can be written as the sum of four parts as follows.

$$H_1(x_1, x_2) = C_{11}x_1 + h_1 E(B_1) \sum_{i=1}^{x_1} (i-1) + h_2 E(B_1) x_1 x_2 + \frac{1}{2} \phi_2 h_2 E(T_1^2(x_1)). \quad (4.26)$$

The first term includes the holding cost incurred by every initial type 1 item during the busy period initiated by itself and the holding cost incurred by all type 1 items arrive during this busy period. The second term is the holding cost incurred by the initial type 1 items before the busy periods initiated by themselves, since the ith initial type 1 item waits for an expected $(i-1)B_1$ amount of time before its service starts, $i = 1, 2, \cdots, x_1$. The third term is the holding cost incurred by x_2 initial type 2 items during $[0, T_1(x_1))$, since the expected waiting time for each of them is $x_1 B_1$. The last term is the holding cost incurred by newly arrived type 2 items during $[0, T_1(x_1))$, since, conditioned on $T_1(x_1) = t$, the expected number of type 2 items arrived during $[0, t)$ is $\phi_2 t$ and the expected waiting time for each of them during $[0, t)$ is $\frac{1}{2}t$.

Note that $T_1(x_1)$ is the busy period for serving type 1 items initiated by x_1 type 1 items. Thus

$$E(T_1^2(x_1)) = Var(T_1(x_1)) + E^2(T_1(x_1)) = x_1 Var(B_1) + (x_1 E(B_1))^2, \quad (4.27)$$

where $E(B_1)$ and $Var(B_1)$ are given by (4.20) and (4.21).

Substituting (4.20), (4.24), (4.25), and (4.27) into (4.26), after some algebra one can show (4.22) holds. $\square$

Lemma 19.

$$\begin{aligned} H_2(x_1, x_2) &= [\phi_1 \phi_2 h_1 \eta^2 \xi + \frac{1}{2} h_2 \xi (2\phi_2 \eta + (\mu + \phi_1)\phi_2^2 \eta^3) + \phi_2^2 h_2 \mu \eta^2 \xi^2] x_1 \\ &+ (\phi_1 h_1 \eta \xi + \frac{1}{2} h_2 \xi + \phi_2 h_2 \mu \eta \xi^2) x_2 + \frac{1}{2} \phi_2^2 h_2 \eta^2 \xi x_1^2 + \phi_2 h_2 \eta \xi x_1 x_2 + \frac{1}{2} h_2 \xi x_2^2. \end{aligned} \quad (4.28)$$

Proof. Because of its lower priority, a type 2 item's service may be interrupted by newly arrived type 1 items. The expected number of interruptions during one service completion time is $\frac{\phi_1}{\mu}$ and each interruption lasts for B_1 amount of time. Hence,

$$C_{12} = \frac{\phi_1}{\mu} h_1 L_{1B} E(B_1). \tag{4.29}$$

Since the service of a type 2 item can only be interrupted by newly arrived type 1 items and items of both types require the same service time, S_2 has the same distribution as B_1. So

$$E(S_2) = E(B_1) = \eta, \tag{4.30}$$

and

$$Var(S_2) = Var(B_1) = (\phi_1 + \mu)\eta^3.$$

Type 2 items view the system as an $M/G/1$ queue with $PP(\phi_2)$ arrival and i.i.d. service times with mean $E(S_2)$ and variance $Var(S_2)$. From Kulkarni [24] (Theorem 7.11), we know

$$L_2 = \rho_2 + \frac{\rho_2^2}{2(1-\rho_2)}(1 + \frac{Var(S_2)}{E^2(S_2)}) = \phi_2\eta + \phi_2^2\mu\eta^2\xi. \tag{4.31}$$

From Prabhu [38] (Chapter 7, Theorem 8), we have

$$E(B_2) = \frac{1}{\mu - \phi_1 - \phi_2}. \tag{4.32}$$

The system state at time $T_1(x_1)$ can be written as $(0, x_2 + K)$, where K is the number of type 2 items arrive during $[0, T_1(x_1))$. For a fixed K, denote by H_{2K} the holding cost incurred by the queueing system starting from state $(0, x_2 + K)$ until state $(0, 0)$ is reached. Assuming LCFS preemptive service discipline within class 2 items, H_{2K} can be written as the sum of three parts as follows.

$$H_{2K} = (x_2+K)\frac{E(B_2)}{E(S_2)}C_{12} + (x_2+K)h_2E(B_2)\frac{L_2}{\rho_2} + h_2E(B_2)\sum_{i=1}^{x_2+K}(i-1). \quad (4.33)$$

The first term is the holding cost incurred by all type 1 items during this period, since $\frac{E(B_2)}{E(S_2)}$ is the expected total number of type 2 items served during B_2. The second term includes the holding cost incurred by every existing type 2 item during the busy period initiated by itself and the holding cost incurred by all type 2 items arrive during this busy period, since $\frac{L_2}{\rho_2}$ is the average number of type 2 items in the system during a busy period initiated by a single type 2 item. The third term is the holding cost incurred by the x_2+K existing type 2 customers before the busy periods initiated by themselves, since the ith existing type 2 customer waits for an expected $(i-1)E(B_2)$ amount of time before its service starts.

Substituting (4.29), (4.30), (4.31), and (4.32) into (4.33), after some algebra, we get

$$\begin{aligned} H_{2K} &= K\phi_1h_1\eta\xi + \frac{1}{2}h_2K(K+1)\xi + Kh_2\phi_2\mu\eta\xi^2 \\ &\quad +[h_1\phi_1\eta\xi + \frac{1}{2}h_2(2K+1)\xi + h_2\phi_2\mu\eta\xi^2]x_2 + \frac{1}{2}h_2\xi x_2^2 \end{aligned} \quad (4.34)$$

For K, the number of type 2 items arrive during the busy period started by x_1 type 1 items, we have

$$E(K) = x_1E(B_1)\phi_2, \quad (4.35)$$

and

$$\begin{aligned} E(K^2) &= E[E(K^2|T_1(x_1))] = E[Var(K|T_1(x_1)) + E^2(K|T_1(x_1))] \\ &= E(\phi_2T_1(x_1) + \phi_2^2T_1^2(x_1)) = \phi_2E(T_1(x_1)) + \phi_2^2E(T_1^2(x_1)). \end{aligned}$$

Plugging in (4.27), we get

$$E(K^2) = [\phi_2\eta + \phi_2^2(\phi_1+\mu)\eta^3]x_1 + \phi_2^2\eta^2 x_1^2. \tag{4.36}$$

Obviously, $H_2(x_1,x_2) = E_K(H_{2K})$. Taking expectation on both sides of (4.34) with respect to K and plugging in (4.35) and (4.36), one can show that (4.28) holds. $\square$

The above results allow us to compute $J(x_1,x_2)$ as given in the following lemma.

Lemma 20.

$$J(x_1,x_2) = Ax_1 + Bx_2 + Cx_1^2 + Dx_1x_2 + Ex_2^2, \tag{4.37}$$

where

$$\begin{aligned}
A &= [c(\phi_1+\phi_2) + \frac{1}{2}h_1]\eta + \phi_1 h_1\eta^2 + [c(\phi_1+\phi_2)\phi_2 + \phi_2 h_2]\eta\xi + \frac{1}{2}(\phi_1+\mu)\phi_2 h_2\eta^3 \\
&\quad +\phi_1\phi_2 h_1\eta^2\xi + \frac{1}{2}(\phi_1+\mu)\phi_2^2 h_2\eta^2\xi^2 + \phi_2^2\mu h_2\eta^2\xi^2, \\
B &= [\frac{1}{2}h_2 + c(\phi_1+\phi_2)]\xi + \phi_1 h_1\eta\xi + \phi_2 h_2\mu\eta\xi^2, \\
C &= \frac{1}{2}(h_1\eta + \phi_2 h_2\eta^2 + \phi_1^2 h_2\eta^2\xi), \\
D &= h_2\eta + \phi_2 h_2\eta\xi, \\
E &= \frac{1}{2}h_2\xi.
\end{aligned}$$

Proof. Denote by $C(x_1,x_2)$ the expected total fixed cost generated by this queueing system starting from initial state (x_1,x_2) until state $(0,0)$ is reached for the first time. Then

$$C(x_1,x_2) = c(\phi_1+\phi_2)[x_1E(B_1) + (x_2+E(K))E(B_2)] = c(\phi_1+\phi_2)[(1+\phi_2\xi)\eta x_1 + \xi x_2]. \tag{4.38}$$

The total cost $J(x_1, x_2)$ can be written as the sum of three parts as follows

$$J(x_1, x_2) = H_1(x_1, x_2) + H_2(x_1, x_2) + C(x_1, x_2). \tag{4.39}$$

Substituting (4.22), (4.28), and (4.38) into (4.39), after some algebra, one can show (4.37) holds. □

We also have

$$\tau(x_1, x_2) = x_1 E(B_1) + (x_2 + E(K))E(B_2).$$

Plugging in (4.20), (4.32), and (4.35), we get

$$\tau(x_1, x_2) = \frac{x_1 + x_2}{\mu - \phi_1 - \phi_2}. \tag{4.40}$$

Substituting (4.16), (4.37), and (4.40) into (4.18) and (4.19), Theorem 15 follows after some algebra. □

4.3 Simulation Study

Although we have the optimal splitting probabilities for the OSI policy and the closed-form expressions for the indices of the GJSQ policy, calculating the expected costs of these policies for the original warranty repair allocation problem is analytically intractable. Therefore, we use simulation to evaluate the performance of these two policies and compare them with two other heuristics. All four heuristics are described below.

- **Join the Shortest Queue policy (JSQ)**: An incoming failed item is sent to the vendor with the shortest repair queue (i.e., the least number of items of both types). If more than one vendor has the shortest queue, among those the item is sent to the one with the smallest fixed cost. In case a tie still exists, it is broken arbitrarily.

- **Optimal State-Independent policy (OSI)**: Incoming failed items are routed according to the optimal Bernoulli splitting probabilities $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ as given in (4.8).

- **Tracking (T)**: An incoming failed type k item is sent to the vendor at which the expected number of type k items under policy $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$ minus the number of existing type k items is the largest, i.e., keep the number of failed items at each vendor as close as possible to the expected number of failed items under the OSI policy.

- **Generalized Join the Shortest Queue policy (GJSQ)**: An incoming failed type k item is sent to the vendor with the smallest type k index. The indices are defined in (4.18) and (4.19).

Our simulation programs were written in SIMSCRIPT II.5, and we use LABATCH.2 (Fishman [14]) to calculate the 95% confidence intervals of the average cost. Each simulation collects data from 2,000 independent replications. Each replication runs for a duration of 5 years and outputs the average yearly cost.

Following Opp, Kulkarni, and Glazebrook [37], we use Gini coefficient (Gini [15]) as a measure of the uniformity of the optimal state-independent allocation, which is connected to the performance of the GJSQ policy. The Gini coefficient is widely used in the economic literature as a measure of income inequality. It is a number between 0 and 1, where 0 corresponds to perfect equality and 1 corresponds to perfect inequality. We use it to measure the inequality of the distribution of repairs among vendors.

The Gini coefficient is calculated using the Lorenz curve (Lorenz [29]), which is a graphical representation of income inequality. In the context of warranty repair allocation, it can be explained as follows. Let x-axis correspond to the percentage of vendors and y-axis correspond to the percentage of repair allocation. The Lorenz curve is a piecewise linear function that contains point (x, y) if the bottom $x\%$ of vendors have $y\%$ of the total repairs (see Figure 4.2). In the case of perfect equality, every vendor gets the same number of repairs and the Lorenz curve becomes the 45° line, which is called the *perfect equality line*. The Gini coefficient is the ratio between the area enclosed by the perfect equality line and the Lorenz curve, and the total area under the perfect equality line.

We illustrate the concepts of Lorenz curve and Gini coefficient using a small warranty repair allocation example. Suppose three vendors provide repair services for two types of items. Sales of type 1 items form a $PP(100)$ and sales of type 2 items form a $PP(300)$. The warranty length and failure rate are the same for both types of items. The optimal state-independent policy is $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*)$, with $\boldsymbol{p}_1^* = (0.2, 0.6, 0.2)$ and $\boldsymbol{p}_2^* = (0.1, 0.5, 0.4)$. We measure the distribution inequality among vendors in terms of total number of repairs (of both types) assigned. Therefore, on average the percentage of total repairs assigned to vendor 1, 2, and 3 are 12.5%, 52.5%, and 35%, respectively. Sorting the vendors in ascending order of repair assignment, we can see that the lowest (33.3%) vendor gets 12.5% of the total assignment, the lowest two (66.7%) vendors get 47.5% of the total assignment, and the lowest three (100%) vendors get 100% of the total assignment. Therefore, the Lorenz curve is a piecewise linear function that connects points $(0, 0), (33.3, 12.5), (66.7, 47.5)$, and $(100, 100)$ as shown in Figure 4.2.

In general, suppose there are V vendors providing repair services for K classes of items. Failures from class k items occur according to $PP(\phi_k)$, $k = 1, 2, \cdots, K$, and the optimal state-independent policy is $(\boldsymbol{p}_1^*, \boldsymbol{p}_2^*, \cdots, \boldsymbol{p}_K^*)$, where $\boldsymbol{p}_k^* = (p_{k1}^*, p_{k2}^*, \cdots, p_{kV}^*)$. Then the Gini coefficient can be calculated using the following formula:

$$G = \frac{\sum_{i=j+1}^{V} \sum_{j=1}^{V} |\sum_{k=1}^{K} \phi_k p_{ki}^* - \sum_{k=1}^{K} \phi_k p_{kj}^*|}{V \sum_{k=1}^{K} \phi_k}.$$

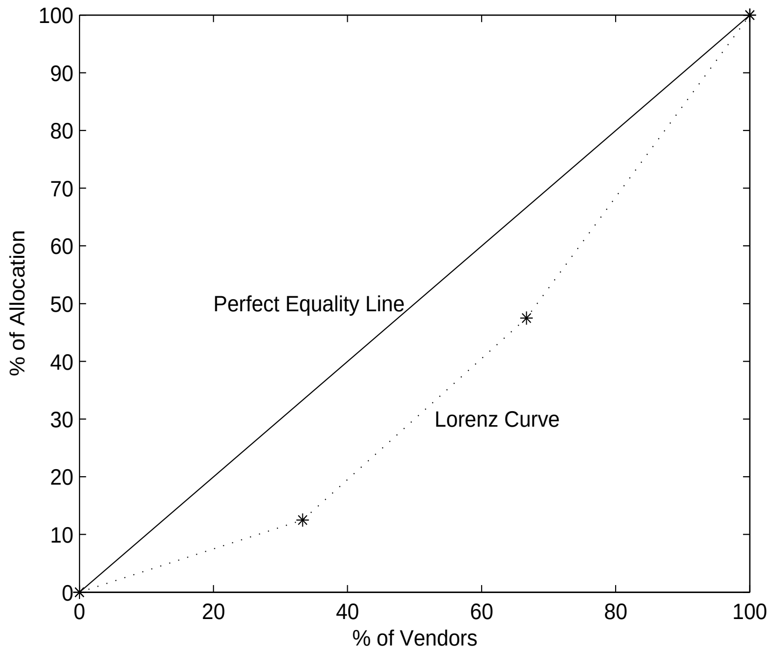

Figure 4.2: Lorenz curve and perfect equality line

Next we present the simulation results as a function of the Gini coefficient of the optimal state-independent allocation.

We simulate a system with 2 types of items and 3 vendors. Sales of each type of items occur according to a Poisson process with rate 200 items per year. Both types of items are covered under warranty for 1 year and have a failure rate of 1.5 failures per item per year. The holding cost rates are the same across all vendors with $h_{1i} = 500, h_{2i} = 300, i = 1, 2, 3$. The fixed cost at each vendor is randomly generated from distribution $U[20, 150]$. A total service rate is randomly generated from distribution $U[605, 1210]$ and is randomly distributed among 3 vendors. Note that the total service rate is guaranteed to be larger than the expected total failure rate, therefore the system is always stable. Since there are only 3 vendors, the Gini coefficient of the optimal state-independent allocation ranges from 0 to $\frac{2}{3}$. 30 random examples are generated for each of the following Gini coefficient ranges: $[0, 0.1), [0.1, 0, 2), \cdots, [0.5, 0.6)$, and 20 random examples are generated for Gini coefficient rage $[0.6, \frac{2}{3}]$. Call cases with Gini coefficient < 0.5 *non-extreme cases*, and those with Gini coefficient ≥ 0.5 *extreme cases*.

Tables 4.2, 4.3, and 4.4 summarize the cost reductions by using the GJSQ policy instead of the other three heuristics. These tables show the minimum, maximum, and mean percent cost reductions among all 200 cases, among the 150 non-extreme cases, and among the 50 extreme cases. As we can see that on average the GJSQ policy performs better than the other heuristics, and there are instances for which the GJSQ policy provides remarkably significant savings over the other heuristics. The cost reduction provided by the GJSQ policy is even larger (except for the maximum and mean reductions over JSQ) when restricted to the non-extreme cases. There are a small number of instances for which the GJSQ policy costs slightly more than the other heuristics, most of which are extreme cases.

Figure 4.3, 4.4, and 4.5 plot the percent cost reductions against Gini coefficients of the optimal state-independent allocations. A straight line is fitted to the data points using the ROBUSTFIT function provided by MATLAB, which uses robust linear regression that is less sensitive to outliers in the data as compared with ordinary least squares regression. Plots 4.4 and 4.5 (corresponding to OSI and T, respectively) show a downward trend in savings as Gini coefficient increases. Plot 4.3 (corresponding to JSQ) shows a slightly upward trend. These observations are consistent with the results summarized in the tables.

Table 4.2 shows the comparison results between the GJSQ policy and the JSQ policy. Among all 200 cases, the GJSQ policy provides an average cost saving of 9.85% over the JSQ policy. Since the JSQ policy tends to allocate items evenly among vendors, it is expected to perform very poorly in extreme cases. In another word, the GJSQ policy has a greater chance to provide large cost reduction over the JSQ policy in extreme cases. This intuition explains the fact that, when restricted to non-extreme cases, the minimum reduction improves but the maximum and mean reductions reduce. Plot 4.3 shows the upward trend although it is not statistically significant. Out of the 150 non-extreme cases, the GJSQ policy provides positive cost reductions in 139 cases. Out of the 50 extreme cases, the GJSQ policy provides positive cost reductions in 20 cases. The negative cost savings are all relatively small (-2.13% in the worst case), while the positive cost savings can be very large (up to 63.5%). As a result, although the GJSQ policy gives negative savings in 60% of the extreme cases, the average cost saving among extreme cases is still positive (14.30%).

	All cases	Cases with Gini coefficient < 0.5	Cases with Gini coefficient ≥ 0.5
Min. reduction	-2.15%	-1.19%	-2.15%
Max. reduction	63.54%	48.83%	63.50%
Mean reduction	9.85%	8.37%	14.30%

Table 4.2: Cost reduction of GJSQ over JSQ

Table 4.3 shows the comparison results between the GJSQ policy and the OSI policy. Among all 200 cases, the GJSQ policy provides an average cost saving of 3.49% over the OSI policy. Among the 150 non-extreme cases, the average cost saving is 4.57%. Plot 4.4 shows the downward trend which is statistically significant at 99% level. The GJSQ policy provides positive cost reductions in all non-extreme cases. Out of the 50 extreme cases, the GJSQ policy provides positive cost reductions in 47 cases. One may argue that the GJSQ policy should never perform worse than the OSI policy, since it improves on top of the optimal state-independent policy by applying a single step of policy improvement. However, when calculating the indices, we ignored the dynamics of the system by assuming the number of functioning items under warranty stays constant and is always the expected number of items under warranty in steady state. This simplifying assumption as well as the error introduced by simulation cause the seemingly lawbreaking behavior.

Table 4.4 shows the comparison results between the GJSQ policy and the T policy. Among all 200 cases, the GJSQ policy provides an average cost saving of 4.23% over

	All cases	Cases with Gini coefficient < 0.5	Cases with Gini coefficient ≥ 0.5
Min. reduction	-3.05%	0.098%	-3.05%
Max. reduction	18.24%	18.24%	7.79%
Mean reduction	3.49%	4.57%	0.24%

Table 4.3: Cost reduction of GJSQ over OSI

the T policy. Among the 150 non-extreme cases, the average cost saving is 4.87%. Plot 4.5 shows the downward trend which is statistically significant at 99% level. The GJSQ policy provides positive cost reductions in all non-extreme cases. Out of the 50 extreme cases, the GJSQ policy provides positive cost reductions in 33 cases.

	All cases	Cases with Gini coefficient < 0.5	Cases with Gini coefficient ≥ 0.5
Min. reduction	-2.16%	0.007%	-2.16%
Max. reduction	21.98%	21.98%	15.65%
Mean reduction	4.23%	4.87%	2.31%

Table 4.4: Cost reduction of GJSQ over T

From the above observations we can see that the GJSQ policy is a very robust and efficient algorithm. It beats the other heuristics on average even when considering only the extreme cases. It can provide significant cost savings over the other heuristics in many cases (up to 63.54% over JSQ, 18.24% over OSI, and 21.98% over T). In the worst case among our 800 random examples, the GJSQ policy costs only 3.05% more.

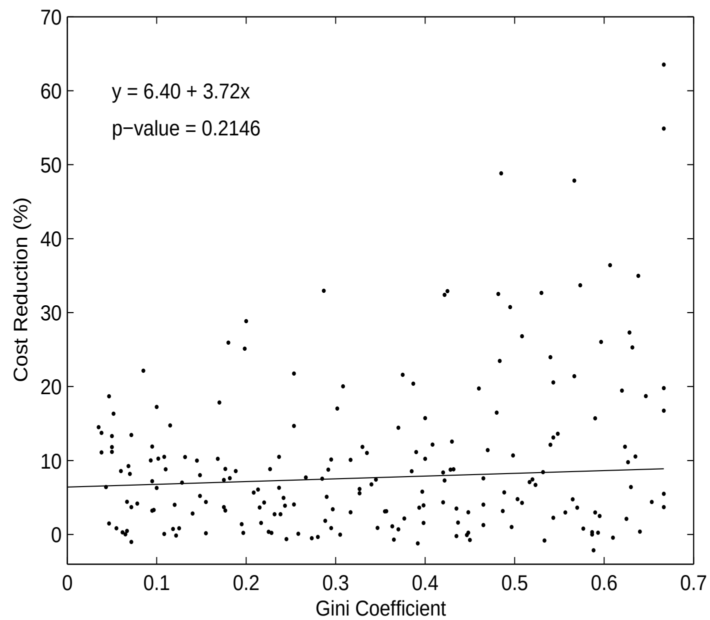

Figure 4.3: Cost reduction of GJSQ over JSQ

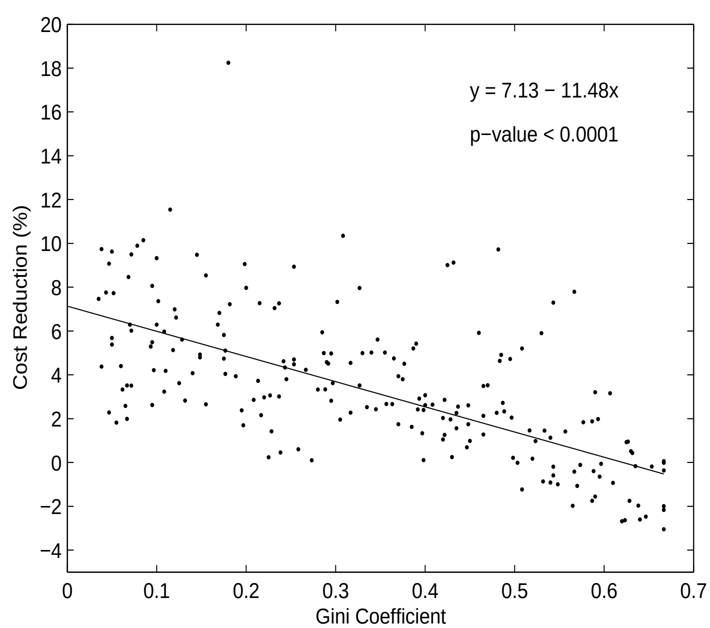

Figure 4.4: Cost reduction of GJSQ over OSI

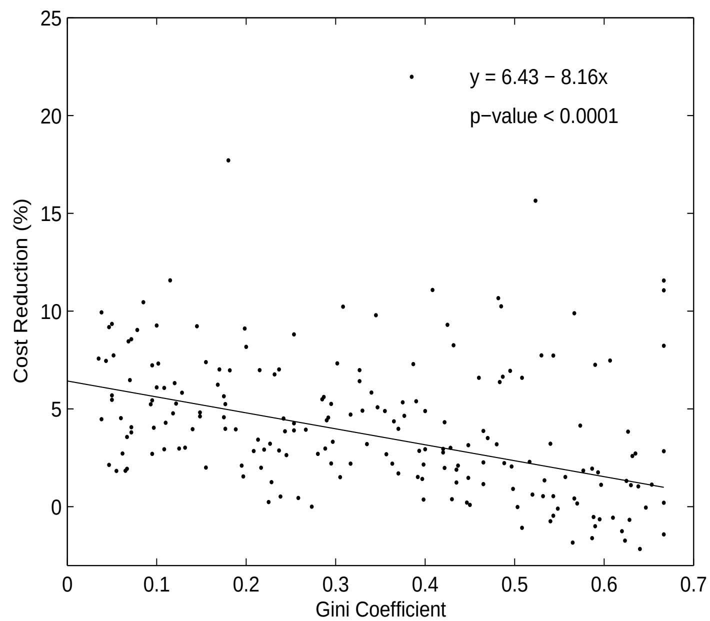

Figure 4.5: Cost reduction of GJSQ over T

Chapter 5

Conclusions and Future Work

5.1 Conclusions

We studied two problems motivated by the prioritized warranty repair outsourcing problem: the admission control problem to a single vendor and the routing problem to multiple vendors.

We considered two cases of the admission control problem. In Chapter 2, we discussed the case where the reward is generated at the time of joinning the repair queue. Modelling the single repair vendor as a two-class $M/M/1$ queueing system with fixed priorities, we analyzed the optimal policies under three criteria, i.e., individual optimization, class optimization, and social optimization, and showed that they are characterized by either critical numbers or monotone switching curves. We also compared different policies and showed that the class-optimal policy accepts more high priority customers and fewer low priority customers than the socially optimal policy. Compared with either socially optimal policy or class-optimal policy, the individually optimal policy accepts more high priority customers, while it may accept either more or fewer low priority customers. In Chapter 3, we considered the case where the reward is generated at the time of service completion. Using sample path argument, we proved that the optimal control policies have the same structural properties as in the first case. We compared different policies numerically. The numerical results suggest the same relationships as in the first case.

In Chapter 4, we addressed the problem of dynamically routing prioritized warranty repairs to multiple vendors. We developed an index-based heuristic which is developed by performing a single step of dynamic programming policy improvement on an optimal state-independent policy. After deriving closed-form expressions for the indices, we evaluated the index policy and compared it with three other heuristics using simulation. The simulation results suggest that the index policy is a robust, efficient algorithm. It can provide a significant cost reduction over the other heuristics, especially when the optimal state-independent allocation is relatively uniform among vendors.

5.2 Future Work

Admission Control

(1) The structural results for the individually optimal policy and the class-optimal policy can be extended to an $M/M/s$ queue with class-dependent service rates by following similar approaches. The extension of the results for the socially optimal policy is more complicated and requires future work.

(2) We have proved the results for the socially optimal policy under assumption $h_1 \geq h_2$. Both intuition and numerical experiments suggest that the results are still true when $h_1 < h_2$. However, it remains to prove them rigorously.

(3) Extend the results to $M/M/1$ queues with more than two priority classes.

(4) Prove the comparison results analytically using sample path argument.

Warranty Repair Routing

(1) Extend the results to problems with n types of items. The algorithm for finding the optimal state-independent policy can be easily extended to n types of items (See Buczkowski et al. [9]) . Unfortunately, generalizing our method to derive the indices requires future work.

(2) If, for some reason, the real-time state information of the vendors is not available, one has to use partially state-dependent policies, i.e., decisions are only based on the real-time information of the warranty population (numbers of items under warranty, and the remaining warranty period for each item). What is a good policy in this case? We expect that an index-based partially state-dependent policy can be developed by following a similar approach.

(3) We have been assuming the repair fees are exogenous and given. We haven't answered the question of how these prices are chosen. Game theoretic models can be used to address this question. Assuming the vendors compete with each other in setting prices and service rates, one can model the competition among vendors as a Nash game and the contracting between the manufacturer and the vendors as a Stackelberg game. The existence (perhaps uniqueness) of the Stackelberg equilibria and Nash equilibria is desirable.

Appendix

The parameters (fixed costs c_1, c_2, c_3 and service rates μ_1, μ_2, μ_3) for the simulation studies in Chapter 4 are given in this appendix.

Table 1: Simulation parameters (Trials 1-25)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
1	138.70	91.99	78.09	432.33	210.43	240.53
2	84.30	73.63	57.70	549.80	223.60	290.45
3	41.13	113.79	93.61	248.33	496.70	216.22
4	116.02	129.28	23.35	313.82	419.20	217.97
5	30.74	41.11	146.88	236.29	200.16	661.62
6	112.68	147.20	105.72	169.30	229.80	222.49
7	50.40	105.09	53.99	252.60	274.23	211.22
8	140.96	25.76	140.03	349.58	227.55	273.39
9	128.51	67.67	36.69	326.78	184.57	275.45
10	122.58	66.70	48.76	432.42	259.27	204.21
11	27.74	80.75	37.72	212.42	467.34	237.25
12	101.53	109.64	39.60	268.95	356.05	254.24
13	79.32	62.83	87.97	318.45	222.77	510.01
14	101.80	79.31	107.17	194.95	213.26	254.51
15	107.19	64.26	60.27	485.51	243.74	189.92
16	40.92	96.23	68.57	291.06	283.35	302.85
17	82.82	147.84	113.85	322.43	220.63	191.33
18	86.41	118.44	32.21	196.12	559.56	181.64
19	43.54	45.92	118.52	159.19	200.01	272.56
20	85.79	36.82	39.01	551.80	360.01	167.79
21	50.91	112.02	135.14	236.32	329.31	197.79
22	128.23	125.42	39.05	321.77	199.66	242.92
23	107.42	41.52	31.89	212.59	288.96	143.27
24	30.01	49.23	122.04	217.02	290.60	376.40
25	43.63	79.93	110.91	319.07	239.52	217.06

Table 2: Simulation parameters (Trials 26-50)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
26	86.73	76.18	101.79	206.71	191.45	302.09
27	20.38	76.10	32.52	241.16	581.60	164.67
28	92.85	145.95	52.15	199.43	568.94	194.44
29	132.05	78.37	99.94	393.25	281.13	286.78
30	22.87	72.27	92.98	282.94	138.63	538.66
31	62.43	39.85	47.46	618.63	356.90	61.27
32	83.47	77.91	82.15	237.20	400.81	411.10
33	113.20	118.75	147.30	345.24	223.64	148.89
34	83.17	127.55	148.96	395.38	120.07	150.80
35	81.31	91.78	140.46	361.85	79.38	412.06
36	120.26	74.98	29.32	731.12	114.62	224.36
37	46.83	98.38	96.99	335.77	145.56	321.80
38	126.46	117.43	96.65	384.19	451.01	177.22
39	38.79	138.69	109.34	46.90	331.97	279.44
40	33.33	110.09	107.73	102.77	196.51	376.90
41	134.90	60.60	98.94	525.89	86.01	331.23
42	85.37	112.23	75.76	384.71	458.54	170.52
43	88.70	56.18	59.73	332.43	76.46	329.47
44	69.17	141.40	127.72	373.75	536.11	118.38
45	123.33	101.15	21.11	397.51	297.61	343.38
46	27.22	33.11	49.86	41.39	291.35	723.54
47	123.51	104.60	37.17	21.92	352.38	294.08
48	95.65	34.14	41.23	309.75	7.87	315.52
49	116.24	141.99	74.31	493.35	70.87	345.91
50	107.44	132.08	87.22	294.23	652.97	28.72

Table 3: Simulation parameters (Trials 51-75)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
51	108.18	87.30	70.33	406.62	3.43	357.65
52	114.05	31.49	73.05	600.08	28.43	250.91
53	34.05	116.85	109.74	357.28	157.53	338.04
54	105.47	63.16	138.65	403.57	27.90	213.96
55	81.77	118.86	30.72	411.82	172.79	209.86
56	120.00	93.57	38.61	149.38	212.10	387.22
57	93.34	138.29	57.14	446.35	353.19	204.66
58	77.20	146.70	107.38	435.00	397.67	62.60
59	149.80	147.49	75.37	7.74	772.33	273.18
60	140.82	99.12	138.39	78.72	191.58	362.58
61	125.13	58.46	88.15	389.88	464.43	17.99
62	97.90	48.61	33.43	245.14	286.59	441.04
63	120.75	127.69	138.07	124.25	449.69	42.70
64	144.73	128.13	56.03	758.27	9.33	242.27
65	25.04	59.54	123.00	491.05	97.93	432.58
66	83.45	41.09	103.22	522.32	493.99	25.02
67	92.19	45.93	62.14	325.66	51.75	527.98
68	78.49	47.09	60.77	41.63	299.09	807.02
69	138.48	106.07	129.16	552.11	442.18	12.77
70	119.13	33.62	105.18	521.09	62.60	142.78
71	136.57	115.93	29.56	982.85	94.95	108.44
72	116.61	147.54	75.70	538.62	3.03	422.76
73	111.36	91.05	77.83	410.45	219.03	478.62
74	103.41	70.14	34.38	192.33	578.70	204.75
75	21.66	140.67	40.04	519.18	218.09	308.55

Table 4: Simulation parameters (Trials 76-100)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
76	102.42	137.01	79.78	149.34	397.56	580.95
77	49.07	41.62	33.22	191.84	635.11	14.57
78	82.92	21.23	73.21	561.07	563.16	32.95
79	137.16	125.20	85.37	65.74	171.40	527.12
80	35.12	25.46	113.06	552.50	29.89	171.04
81	32.50	41.73	69.67	598.89	288.57	73.98
82	33.31	78.26	118.36	25.75	115.35	977.76
83	110.8	136.53	34.63	224.06	80.96	520.11
84	72.16	95.04	82.82	590.29	405.04	101.00
85	114.79	83.10	92.83	481.99	578.85	117.83
86	21.55	68.40	66.05	83.47	751.78	49.65
87	49.41	141.23	116.54	503.98	249.35	431.86
88	28.31	84.59	80.04	105.71	104.18	882.37
89	108.91	139.89	145.83	599.21	139.36	50.27
90	34.24	27.92	110.95	665.24	201.24	4.16
91	81.07	50.77	127.61	37.54	636.92	121.78
92	39.64	48.33	73.16	637.19	15.54	42.07
93	26.60	68.72	136.51	607.62	126.4	108.59
94	27.96	51.28	63.68	20.98	19.65	779.34
95	56.59	140.63	39.72	29.41	29.08	702.77
96	20.35	59.22	30.87	6.65	149.91	764.46
97	83.87	129.58	67.81	277.73	208.13	664.11
98	29.71	128.20	57.19	53.50	187.08	774.19
99	23.12	109.97	130.09	775.91	198.74	118.43
100	37.15	98.41	121.58	54.10	852.23	52.73

Table 5: Simulation parameters (Trials 101-125)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
101	131.80	130.98	97.16	263.17	222.24	210.12
102	26.00	59.62	125.80	228.34	197.19	668.94
103	114.04	51.17	83.68	607.03	179.16	288.71
104	115.39	98.31	27.40	420.51	290.69	204.42
105	97.67	109.08	146.60	289.62	199.91	305.51
106	39.39	105.91	50.80	253.45	248.19	165.03
107	62.46	54.43	86.34	241.51	155.06	219.12
108	23.67	99.50	88.11	235.01	170.00	211.45
109	24.43	61.52	145.02	248.15	176.84	238.78
110	36.03	86.62	32.50	208.45	269.46	195.49
111	103.61	24.87	136.31	280.88	191.04	549.23
112	54.83	125.18	73.67	253.94	687.56	227.10
113	69.56	75.14	84.67	229.10	305.50	583.73
114	52.39	145.35	65.21	255.82	513.22	254.89
115	50.18	79.91	24.60	250.17	554.16	258.16
116	92.74	71.70	142.94	323.29	228.38	395.46
117	60.62	137.48	110.36	332.40	225.44	202.03
118	104.48	20.87	21.43	184.34	234.90	308.81
119	117.55	46.75	32.32	665.92	132.39	321.64
120	24.03	33.53	27.95	185.03	353.34	320.69
121	93.01	147.68	123.83	282.25	209.88	138.95
122	21.64	94.05	80.04	259.18	625.49	191.23
123	138.98	83.84	117.29	385.59	324.66	156.60
124	148.11	75.87	134.39	367.62	112.02	301.88
125	146.96	33.67	66.61	272.84	210.26	158.89

Table 6: Simulation parameters (Trials 126-150)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
126	73.44	113.74	115.01	168.56	520.73	369.77
127	62.00	98.58	50.66	181.47	643.10	341.65
128	146.76	77.16	62.94	291.20	274.86	175.70
129	75.13	91.23	119.52	157.21	278.75	706.91
130	143.82	65.38	108.12	223.55	147.69	264.07
131	80.69	89.27	100.12	159.38	409.84	305.55
132	132.32	79.35	112.64	174.02	254.09	352.56
133	142.32	52.28	109.82	413.63	105.59	385.22
134	67.67	77.84	63.06	56.62	297.85	336.12
135	118.41	99.98	37.64	96.93	202.63	360.91
136	124.97	139.99	116.66	195.08	632.53	158.81
137	48.26	105.69	87.68	296.31	263.03	69.95
138	53.90	120.28	136.79	244.84	129.16	739.33
139	68.06	66.24	69.79	167.03	453.56	347.27
140	139.29	60.07	112.45	625.75	95.74	312.71
141	31.66	83.84	110.00	322.72	290.42	206.33
142	22.90	81.41	122.07	256.36	114.56	397.06
143	114.09	50.90	23.95	289.44	390.23	154.07
144	108.15	127.22	70.97	374.57	333.78	240.01
145	31.09	125.56	27.70	156.54	425.36	146.39
146	93.36	88.09	52.70	523.42	108.49	197.03
147	22.28	145.84	129.62	380.59	93.99	726.55
148	27.66	35.44	135.56	31.16	263.37	554.11
149	107.97	25.56	138.87	374.90	30.29	240.64
150	109.96	32.20	61.68	202.91	419.38	130.56

Table 7: Simulation parameters (Trials 151-175)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
151	106.05	141.84	146.34	459.88	281.08	348.45
152	56.36	129.62	58.57	89.24	453.24	89.22
153	100.94	146.47	146.90	279.73	624.37	253.17
154	21.07	123.24	37.08	89.92	461.63	158.50
155	84.89	78.83	101.97	344.50	450.06	242.49
156	25.08	27.99	57.36	77.30	214.69	652.39
157	121.12	76.00	64.40	386.67	464.50	98.64
158	123.86	52.97	23.82	150.33	401.00	95.90
159	92.05	130.37	32.93	378.69	83.68	378.46
160	115.27	84.44	83.84	0.67	382.55	284.76
161	45.62	147.01	136.87	222.66	111.46	543.99
162	35.05	133.98	132.30	129.86	339.69	723.83
163	57.07	32.67	141.14	177.25	4.47	480.82
164	52.54	63.48	123.85	488.04	657.17	50.91
165	60.64	120.05	121.64	182.67	816.70	68.99
166	135.92	81.23	26.23	131.26	17.41	461.42
167	123.99	79.51	37.88	72.27	485.41	504.78
168	83.59	89.08	57.38	48.64	216.72	497.02
169	86.63	102.64	117.01	484.20	138.29	42.50
170	24.92	57.24	72.32	225.76	766.39	59.98
171	82.14	22.61	89.60	78.46	505.42	576.34
172	93.81	68.96	25.73	668.37	76.24	134.25
173	119.24	52.95	135.48	97.76	53.79	465.82
174	44.51	25.04	140.74	456.00	50.51	140.33
175	94.10	146.10	110.56	540.25	279.26	360.44

Table 8: Simulation parameters (Trials 176-200)

Trial	c_1	c_2	c_3	μ_1	μ_2	μ_3
176	69.04	61.73	91.98	539.96	183.18	386.49
177	146.32	39.77	82.64	929.24	44.29	27.73
178	102.97	95.83	92.02	131.30	233.89	639.96
179	135.66	125.52	105.14	409.75	681.27	80.54
180	41.18	104.35	96.93	582.34	10.45	134.83
181	144.22	114.56	75.12	126.67	144.18	542.19
182	23.13	138.35	41.32	17.11	723.85	63.99
183	44.02	23.05	137.94	110.26	565.65	98.31
184	129.53	35.76	62.55	273.84	80.15	594.45
185	78.05	83.83	37.40	960.44	140.20	97.99
186	78.64	89.01	135.78	566.76	162.18	451.87
187	143.25	97.90	54.61	54.25	422.81	546.85
188	72.54	103.44	57.01	814.96	85.90	193.26
189	130.21	109.42	130.12	374.15	599.27	187.69
190	72.07	38.06	101.09	568.69	32.01	75.29
191	23.57	110.36	54.93	36.65	396.08	628.03
192	125.50	24.03	108.81	280.60	763.48	77.31
193	103.84	126.34	105.83	741.87	70.56	90.50
194	143.51	138.03	122.57	51.45	89.54	962.01
195	135.75	87.02	103.39	129.10	65.34	924.16
196	94.20	59.74	62.17	208.13	702.09	100.76
197	108.86	122.93	103.68	34.60	290.52	859.97
198	38.22	22.81	81.42	94.36	670.96	273.41
199	103.56	90.82	109.62	984.01	94.94	31.71
200	63.63	144.13	122.21	607.90	13.74	40.62

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